1974

Electron radial wave functions in beta decay and high energy electron scattering with a diffused nuclear charge distribution.

Juhachi. Asai
University of Windsor

ELECTRON RADIAL WAVE FUNCTIONS IN BETA DECAY

AND HIGH ENERGY ELECTRON SCATTERING

WITH A DIFFUSED NUCLEAR

CHARGE DISTRIBUTION

by

Juhachi Asai

A Dissertation
Submitted to the Faculty of Graduate Studies through the
Department of Physics in Partial Fulfillment
of the Requirements for the Degree of
Doctor of Philosophy at the
University of Windsor

Windsor, Ontario

1974

ABSTRACT

It has been known for many years that the analysis of beta-decay experiments requires an accurate knowledge of the electron radial wave functions. This has been met by considering the finite nuclear size effect[1], the finite deBroglie wavelength effect[2] and the screening effect[3] due to the atomic electrons. The first two effects were taken into account by Bhalla and Rose[4,5] (BR) and they tabulated the values of the electron wave functions evaluated at the surface of a uniformly charged nucleus. However these wave functions do not approach the values for a point nucleus when the nuclear radius is made very small, and it has been believed[6,7] that their tabulation might be in error for positron emission, while it is correct for electron emission.

In the first part of the study, the electron wave functions used in the beta-decay are discussed. In order to re-examine the BR method, we introduce a spherically symmetric diffused charge distribution which tends to the uniform charge distribution as the thickness of the diffused surface of a nucleus approaches zero. The potential associated with the proposed charge distribution is calculated analytically. Using this potential the electron radial wave functions are calculated and compared with the conventional wave functions derived by assuming a point charge and the uniformly charged nucleus.

In the second part of this study, the high energy electron scattering by a nucleus is investigated. Since the representation commonly used in the analysis of the high energy electron scattering is different from that of BR, a method is developed in order to apply the theory in Part I to the analysis of the electron scattering. Although the electron mass is customarily neglected in the high energy electron scattering, we have included the mass explicitly which does or does not affect the scattering analysis, depending on the energy of the incident electrons. Using the diffused charge distribution, the scattering cross section is calculated with the help of the method of phase shift analysis.

ACKNOWLEDGEMENTS

I should like to express my sincere gratitude to Dr. H. Ogata for his supervision and support of this work. It was he who initiated my enthusiasm for nuclear physics.

I should also like to express my gratitude to Dr. M. Soga for many stimulating and informative discussions.

I am particularly indebted to Dr. A. Fujii for his constant interest during the course of this work.

To my colleagues and staff of the Physics department with whom I have been associated over the years I express my thanks for their various kinds of help, particularly to Mr. J. P. Labrie for his assistance of programming at the early stage of CALCO 2 plot.

To the Department of Physics I am grateful for financial support, and to Assumption University for financial assistance in the form of scholarship.

Finally it is a pleasure to record my gratitude to my wife Sumiko for her encouragement and for patient typing of this manuscript.

CONTENTS

		Page
ABSTRACT		ii
ACKNOWLEDGEMENTS		iv
PART I	ELECTRON RADIAL WAVE FUNCTIONS AT NUCLEAR SURFACE	1
CHAPTER 1	GENERAL DESCRIPTIONS FOR A RELATIVISTIC ELECTRON	2
CHAPTER 2	COULOMB RADIAL WAVE FUNCTION AND ITS DEVIATION DUE TO THE NUCLEAR FINITE SIZE EFFECT	10
CHAPTER 3	DIFFUSED NUCLEAR CHARGE DISTRIBUTION	30
CHAPTER 4	DEPENDENCE ON THE ELECTRON RADIAL WAVE FUNCTIONS IN BETA DECAYS	43
CHAPTER 5	DISCUSSION AND CONCLUSION	54
PART II	HIGH ENERGY ELECTRON SCATTERING	67
CHAPTER 6	ELASTIC ELECTRON SCATTERING	68
CHAPTER 7	RELATIONS BETWEEN PHASES USED IN BETA DECAYS AND ELECTRON SCATTERING	89
CHAPTER 8	DISCUSSION AND CONCLUSION	101

APPENDICES		107
APPENDIX A	CHARGE DISTRIBUTION AND POTENTIAL	108
APPENDIX B	ANGULAR MOMENTUM AND DIRAC MATRICES	112
APPENDIX C	EIGENVALUE OF $\sigma_r \chi_{\kappa\mu}$	117
APPENDIX D	POWER SERIES SOLUTIONS FOR THE "INSIDE" REGION	121
APPENDIX E	GUDERMANNIAN FUNCTION AND HYPERBOLIC FUNCTION	130
APPENDIX F	NORMALIZATION CONDITION OF A DIFFUSED CHARGE DISTRIBUTION	132
APPENDIX G	POTENTIAL ASSOCIATED WITH DIFFUSED CHARGE DISTRIBUTION	136
APPENDIX H	POWER SERIES SOLUTIONS FOR THE "INTERMEDIATE" REGION	142
APPENDIX I	ROOT MEAN-SQUARE RADIUS FOR DIFFUSED CHARGE DISTRIBUTION	149
APPENDIX J	RECURRENCE RELATIONS FOR THE LEGENDRE FUNCTIONS	151
APPENDIX K	ALTERNATIVE METHOD TO DETERMENE η IN THE HIGH ENERGY LIMIT	155
APPENDIX L	COULOMB PHASE SHIFT FOR THE IRREGULAR SOLUTION	157
BIBLIOGRAPHY		162

PART I

ELECTRON RADIAL WAVE FUNCTIONS AT NUCLEAR SURFACE

CHAPTER 1

GENERAL DESCRIPTIONS FOR A RELATIVISTIC ELECTRON

A Dirac equation of an electron in a spherically symmetric potential V(r) can be written as[8,9)

$$H\Psi = W\Psi = \left[-(\vec{\alpha}\cdot\vec{p} + \beta) + V(r)\right]\Psi \qquad (1\text{-}1)$$

where we have used the relativistic rationalized units, i.e., $\hbar = c = m$ (electron mass) $= 1$, and 4-by-4 matrices $\vec{\alpha}$ and β can be represented by 2-by-2 Pauli matrices and the unit matrix as

$$\vec{\alpha} = \begin{pmatrix} 0 & \vec{\sigma} \\ \vec{\sigma} & 0 \end{pmatrix}, \qquad \beta = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}.$$

For the case of a point nucleus with charge Ze, we have the Coulomb potential

$$V(r) = -\frac{Ze^2}{r} = -\frac{\alpha Z}{r} \qquad (1\text{-}2)$$

since in the relativistic rationalized units, the fine structure constant α is found to be

$$\alpha = \frac{e^2}{\hbar c} = e^2 \quad .$$

If we assume, on the other hand, that a nucleus has a finite size ρ, called a nuclear radius, and that the density distribution for the nuclear charge is ρe, the potential associated with the assumed charge distribution, in general, can be obtained by the integral

$$V(r) = -4\pi Ze^2\left[\frac{1}{r}\int_0^r \rho_e(r')r'^2dr' + \int_r^\infty \rho_e(r')r'dr'\right] \qquad (1\text{-}3)$$

where the normalization has been chosen such that

$$4\pi\int_0^\infty \rho_e(r')r'^2dr' = 1 . \qquad (1\text{-}4)$$

Now if the uniform charge distribution is assumed, we may find the potential

$$V(r) = -\frac{\alpha Z}{2\rho}\left(3-\frac{r^2}{\rho^2}\right) \quad \text{for} \quad r < \rho . \qquad (1\text{-}5)$$

The derivation of eq.(1-5) is given in Appendix A. The nuclear radius ρ can be determined by the expression

$$\rho = 1.2A^{1/3} \text{ fermi}$$

where A is a mass number of a nucleus. The Compton wavelength of the electron is known to be

$$\lambda = \hbar c/mc^2 = 3.861592 \times 10^{-11} \text{ cm} \quad 10)$$
$$= 3.861592 \times 10^2 \text{ fermi} .$$

Thus one fermi corresponds to 2.589605×10^{-3} in the relativistic rationalized unit.

Equation (1-1) will now be rewritten in terms of spherical coordinates which are appropriate to a central potential. This can be done as follows. The gradient operator is expressed in polar coordinates as

$$\vec{\nabla} = \hat{r}\frac{\partial}{\partial r} - i\frac{\hat{r}}{r}\times\vec{L} .$$

With this identity, we have

$$\vec{\alpha}\cdot\vec{\nabla} = \alpha_r \frac{\partial}{\partial r} - \frac{i}{r}\vec{\alpha}\cdot\hat{r}\times\vec{L} \tag{1-6a}$$

where the radial component of $\vec{\alpha}$ is designated by α_r. Now using the general relation for any vectors $\vec{A}$ and $\vec{B}$,

$$(\vec{\sigma}\cdot\vec{A})(\vec{\sigma}\cdot\vec{B}) = \vec{A}\cdot\vec{B} + i\vec{\sigma}\cdot(\vec{A}\times\vec{B})$$

eq. (1-6a) can be reduced to a simpler form

$$\vec{\alpha}\cdot\vec{\nabla} = \alpha_r\left(\frac{\partial}{\partial r} - \frac{1}{r}\vec{\sigma}\cdot\vec{L}\right). \tag{1-6b}$$

Full derivations can be found in the Appendix B.

We now introduce the wave function of the form

$$\Psi = \begin{pmatrix} -if\chi_{-\kappa\mu} \\ g\chi_{\kappa\mu} \end{pmatrix} \tag{1-7}$$

where f and g are radial wave functions and -i is multiplied for f as a matter of convenience to make both f and g be real. As will be shown later, the symbol κ stands for both the total angular momentum j and the orbital angular momentum l according to the expressions

$$j = |\kappa| - 1/2,$$

$$l = \kappa \quad \text{if } \kappa > 0,$$

and

$$l = -(\kappa+1) \quad \text{if } \kappa < 0.$$

We note that κ is a positive or negative integer excluding the value zero. The symbol μ desighates the eigenvalue of the third component of the total angular momentum. Table 1.1 shows the relation between the total angular momentum j and its associated orbital angular momentum l together with κ (indicated in the bracket).

Table 1.1

The values of j with corresponding l and κ.

j \ κ, l	$l=j-1/2$	$(\kappa<0)$	state	$l=j+1/2$	$(\kappa>0)$	state
1/2	0	(-1)	$s_{1/2}$	1	(1)	$p_{1/2}$
3/2	1	(-2)	$p_{3/2}$	2	(2)	$d_{3/2}$
5/2	2	(-3)	$d_{5/2}$	3	(3)	$f_{5/2}$
7/2	3	(-4)	$f_{7/2}$	4	(4)	$g_{7/2}$

From eq.(1-1) with eq.(1-6a), we can show that the Dirac equation in polar coordinates takes a form

$$(W - V)\Psi = i\alpha_r\left(\frac{\partial}{\partial r} - \frac{1}{r}\vec{\sigma}\cdot\vec{L}\right)\Psi - \beta\Psi . \qquad (1\text{-}8)$$

Inserting eq.(1-7) into eq.(1-8) yields

$$(W - V)\begin{pmatrix}-if\chi_{-\kappa\mu}\\ g\chi_{\kappa\mu}\end{pmatrix}$$

$$= i\alpha_r\left(\frac{\partial}{\partial r} - \frac{1}{r}\vec{\sigma}\cdot\vec{L}\right)\begin{pmatrix}-if\chi_{-\kappa\mu}\\ g\chi_{\kappa\mu}\end{pmatrix} - \begin{pmatrix}I & 0\\ 0 & -I\end{pmatrix}\begin{pmatrix}-if\chi_{-\kappa\mu}\\ g\chi_{\kappa\mu}\end{pmatrix}$$

$$= i\alpha_r\begin{pmatrix}-if'\chi_{-\kappa\mu}\\ g'\chi_{\kappa\mu}\end{pmatrix} - \frac{i}{r}\alpha_r\vec{\sigma}\cdot\vec{L}\begin{pmatrix}-if\chi_{-\kappa\mu}\\ g\chi_{\kappa\mu}\end{pmatrix} - \begin{pmatrix}I & 0\\ 0 & -I\end{pmatrix}\begin{pmatrix}-if\chi_{-\kappa\mu}\\ g\chi_{\kappa\mu}\end{pmatrix}$$

$$= i\begin{pmatrix} \sigma_r g' \chi_{\kappa\mu} \\ -i\sigma_r f' \chi_{-\kappa\mu} \end{pmatrix} - \frac{i}{r}\begin{pmatrix} g\sigma_r \vec{\sigma}\cdot\vec{L}\,\chi_{\kappa\mu} \\ -if\sigma_r \vec{\sigma}\cdot\vec{L}\,\chi_{-\kappa\mu} \end{pmatrix} + \begin{pmatrix} if\chi_{-\kappa\mu} \\ g\chi_{\kappa\mu} \end{pmatrix} \tag{1-9}$$

where the prime means the differentiation with respect to r. Now we have to evaluate $\vec{\sigma}\cdot\vec{L}\,\chi_{\kappa\mu}$ and $\sigma_r\chi_{\kappa\mu}$. The former can be easily evaluated in the following way. Remembering that $l = \kappa$ and $l = -\kappa-1$ for $\kappa>0$ and for $\kappa<0$ respectively,

$$\vec{S}\cdot\vec{L}\ \chi_{\kappa\mu}$$

$$= \frac{1}{2}\left[\vec{J}^2 - \vec{L}^2 - \vec{S}^2\right]\chi_{\kappa\mu}$$

$$= \frac{1}{2}\left[j(j+1)-l(l+1)-s(s+1)\right]\chi_{\kappa\mu}$$

$$= \begin{cases} \frac{1}{2}\left[(|\kappa|-\frac{1}{2})(|\kappa|+\frac{1}{2}) - \kappa(\kappa+1) - \frac{3}{4}\right]\chi_{\kappa\mu} & \text{for } \kappa>0 \\ \frac{1}{2}\left[(|\kappa|-\frac{1}{2})(|\kappa|+\frac{1}{2}) - \kappa(\kappa+1) - \frac{3}{4}\right]\chi_{\kappa\mu} & \text{for } \kappa<0 \end{cases}$$

$$= -\frac{1}{2}(\kappa+1)\chi_{\kappa\mu} \qquad \text{for } \kappa \lessgtr 0 .$$

Therefore we have

$$\vec{\sigma}\cdot\vec{L}\,\chi_{\kappa\mu} = 2\vec{S}\cdot\vec{L}\,\chi_{\kappa\mu} = -(\kappa+1)\chi_{\kappa\mu}$$

or equivalently

$$(1+\vec{\sigma}\cdot\vec{L})\chi_{\kappa\mu} = -\kappa\chi_{\kappa\mu} . \tag{1-10a}$$

The latter, $\sigma_r\chi_{\kappa\mu}$, may be obtained by considering the fact that σ_r is a scalar product of two first-rank tensor $\hat{r}$ and $\vec{\sigma}$. This turns out to be

$$\sigma_r\chi_{\kappa\mu} = -\chi_{-\kappa\mu} . \tag{1-10b}$$

(See Appendix C.)

From eqs.(1-9), (1-10a) and (1-10b), we can write schematically

$$(W - V)\begin{pmatrix} -if\chi_{-\kappa\mu} \\ g\chi_{\kappa\mu} \end{pmatrix} = \begin{pmatrix} -ig' - i(\kappa+1)g/r + if \\ -f' + (\kappa-1)f/r + g \end{pmatrix}\begin{pmatrix} \chi_{-\kappa\mu} \\ \chi_{\kappa\mu} \end{pmatrix}.$$

Thus

$$-i\chi_{-\kappa\mu}\left[(W - V + 1)f - g' - \frac{\kappa+1}{r}g \right] = 0 ,$$

$$\chi_{\kappa\mu}\left[(W - V - 1)g + f' - \frac{\kappa-1}{r}f \right] = 0 .$$

Finally we see that the real radial wave functions f and g satisfy the coupled first order differential equations.

$$\frac{df}{dr} = \frac{\kappa-1}{r}f - (W - V - 1)g , \qquad (1\text{-}11a)$$

$$\frac{dg}{dr} = -\frac{\kappa+1}{r}g + (W - V + 1)f . \qquad (1\text{-}11b)$$

For convenience, we define F and G as

$$F = rf , \qquad (1\text{-}12a)$$

$$G = rg . \qquad (1\text{-}12b)$$

We have, then, instead of eqs.(1-11a) and (1-11b)

$$\frac{dF}{dr} = \frac{\kappa}{r}F - (W - 1 - V)G , \qquad (1\text{-}13a)$$

$$\frac{dG}{dr} = -\frac{\kappa}{r}G + (W + 1 - V)F . \qquad (1\text{-}13b)$$

Before going into the next discussion it might be useful to introduce an operator K defined as

$$K = \beta(\vec{\sigma}\cdot\vec{L} + 1) \qquad (1\text{-}14)$$

which satisfies the eigenvalue equation,

$$K\Psi = \beta(\vec{\sigma}\cdot\vec{L} + 1)\Psi = \kappa\Psi . \tag{1-15}$$

In fact, since β and σ commute and $\beta^2 = 1$, we have, using eq.(10-a),

$$\beta^2(\vec{\sigma}\cdot\vec{L} + 1)\chi_{\kappa\mu} = -\kappa\chi_{\kappa\mu}$$

which is equal to

$$\beta K \chi_{\kappa\mu} = -\kappa\chi_{\kappa\mu}.$$

Therefore it is found that

$$K\Psi = \beta(\vec{\sigma}\cdot\vec{L} + 1)\begin{pmatrix} -if\chi_{-\kappa\mu} \\ g\chi_{\kappa\mu} \end{pmatrix}$$

$$= \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}\begin{pmatrix} -i\kappa f\chi_{-\kappa\mu} \\ -\kappa g\chi_{\kappa\mu} \end{pmatrix}$$

$$= \kappa\begin{pmatrix} -if\chi_{-\kappa\mu} \\ g\chi \end{pmatrix} = \kappa\Psi . \tag{1-16}$$

The operator K is connected to $\vec{j}$ in the following manner: Using the relation

$$(\vec{\sigma}\cdot\vec{L})(\vec{\sigma}\cdot\vec{L}) = \vec{L}^2 + i\vec{\sigma}\cdot(\vec{L}\times\vec{L})$$

$$= \vec{L}^2 - \vec{\sigma}\cdot\vec{L} ,$$

one can readily see that

$$K^2 = 1 + \vec{\sigma}\cdot\vec{L} + \vec{L}^2 .$$

Since $\vec{j} = \vec{L} + \vec{S} = \vec{L} + \frac{\vec{\sigma}}{2}$,

$$\vec{j}^2 = \vec{L}^2 + \vec{\sigma}\cdot\vec{L} + \frac{3}{4} ,$$

hence $$K^2 = \vec{j}^2 + \frac{1}{4} \; . \tag{1-17}$$

Calling κ^2 the eigenvalue of K^2, we have

$$\kappa^2 = j(j+1) + \frac{1}{4} = (j + \frac{1}{2})^2 \; . \tag{1-18}$$

CHAPTER 2

COULOMB RADIAL WAVE FUNCTION AND ITS DEVIATION DUE TO THE NUCLEAR FINITE SIZE EFFECT

In the preceding chapter, we laid the basic differential equations for the electron radial wave functions. In this chapter we further develop the theory to determine the functional form of the electron wave functions. First we discuss a point nucleus. Since even the most energetic electrons in the beta-decay have wavelength which are long compared to nuclear dimensions, it is appropriate, as a first step, to treat a nucleus as a point charge. Secondly we discuss the extended nucleus with the uniform charge distribution. Lastly we describe a method through which the actual calculation is performed by expanding the electron radial wave functions in powers of the radial variable.

We denote the Coulomb wave functions by F^c and G^c, which satisfy the coupled first-order differencial equations

$$\frac{dF^c}{dr} = \frac{\kappa}{r}F^c - (W - 1 - V)G^c \quad , \tag{2-1}$$

$$\frac{dG^c}{dr} = - \frac{\kappa}{r}G^c + (W + 1 - V)F^c \tag{2-2}$$

where

$$V(r) = - \frac{\alpha Z}{r} \quad .$$

To obtain these wave functions, we set[11)]

$$F^c = i(W - 1)^{\frac{1}{2}}(\varphi_1 - \varphi_2) \tag{2-3}$$

$$G^c = (W + 1)^{\frac{1}{2}}(\varphi_1 + \varphi_2) \tag{2-4}$$

Introducing a new variable x as 2ipr, we obtain the following differential equations for φ_1 and φ_2 from eqs.(2-1) and (2-2)

$$d\varphi_1/dx = \left(\frac{1}{2} + i\frac{\alpha ZW}{px}\right)\varphi_1 - \left(\frac{\kappa}{x} - \frac{i\alpha Z}{px}\right)\varphi_2 , \tag{2-5}$$

$$d\varphi_2/dx = -\left(\frac{\kappa}{x} + \frac{i\alpha Z}{px}\right)\varphi_1 - \left(\frac{1}{2} + i\frac{\alpha ZW}{px}\right)\varphi_2 , \tag{2-6}$$

where $p = (W^2 - 1)^{\frac{1}{2}}$.

If we take the complex conjugate of these equations, eq.(2-6), say, keeping in mind that x is a pure imaginary quantity, we find

$$d\varphi_2^*/dx = -\left(\frac{\kappa}{x} - \frac{i\alpha Z}{px}\right)\varphi_1^* + \left(\frac{1}{2} + i\frac{\alpha ZW}{px}\right)\varphi_2^* . \tag{2-7}$$

Comparing eq.(2-7) with eq.(2-5), we find that these equations are identical if we set

$$\varphi_1 = \varphi_2^* \quad \text{or} \quad \varphi_2 = \varphi_1^* . \tag{2-8}$$

As is obviously seen from the original equations (2-3) and (2-4) and from eq.(2-8),

$$F^{c*} = -i(W - 1)^{\frac{1}{2}}(\varphi_1^* - \varphi_2^*)$$

$$= i(W - 1)^{\frac{1}{2}}(\varphi_1 - \varphi_2)$$

$$= F^c$$

similarly we have $G^{c*} = G^c$. Hence F^c and G^c are chosen to be real. To proceed further, we eliminate φ_2 in order to get a second-order differential equation for φ_1. This leads to the equation

$$\frac{d^2\varphi_1}{dx^2} + \frac{1}{x}\frac{d\varphi_1}{dx} - \left[\frac{1}{4} + \frac{1}{x}\left(\frac{1}{2} + i\frac{\alpha ZW}{p}\right) + \frac{\gamma^2}{x^2}\right]\varphi_1 = 0$$

where

$$\gamma = (\kappa^2 - (\alpha Z)^2)^{\frac{1}{2}} \quad (2\text{-}9)$$

To write this in normal form, we again define

$$\Phi = x^{\frac{1}{2}}\varphi_1 \quad (2\text{-}10)$$

and get a differential equation for Φ.

$$\frac{d^2\Phi}{dx^2} - \left[\frac{1}{4} + \frac{1}{x}\left(\frac{1}{2} + iy\right) + \frac{\gamma^2 - 1/4}{x^2}\right]\Phi = 0 \quad (2\text{-}11)$$

where

$$y = \frac{\alpha ZW}{p} \quad (2\text{-}12)$$

This differential equation is called Whittaker's equation.[12)]

It has the solution

$$\Phi_1 = x^{\gamma+1/2} e^{-x/2} {}_1F_1(\gamma+1+iy, 2\gamma+1; x) \qquad (2\text{-}13a)$$

$$\Phi_2 = x^{-\gamma+1/2} e^{-x/2} {}_1F_1(-\gamma+1+iy, -2\gamma+1; x) \qquad (2\text{-}13b)$$

${}_1F_1$ is the confluent hypergeometric function which is given by

$${}_1F_1(a, b; z) = F(a, b; z)$$

$$= 1 + \frac{a}{b} z + \frac{a(a+1)}{b(b+1)} \frac{z^2}{2!} + \frac{a(a+1)(a+2)}{b(b+1)(b+2)} \frac{z^3}{3!} + \text{-----}$$

$$= \frac{\Gamma(b)}{\Gamma(a)} \sum_{m=0}^{\infty} \frac{\Gamma(a+m)}{\Gamma(b+m)} \frac{z^m}{m!} .$$

As is easily seen from eqs.(2-10), (2-13), Φ_1 corresponds to the solution regular at $r = 0$ and Φ_2 the irregular solution. Thus, to obtain the irregular solution, we simply replace γ in expression (2-13a) for Φ_1 with $-\gamma$. The general solution at $r \neq 0$ is a linear combination of the regular and irregular solutions. However, understanding that the irregular solution can be obtained from the regular solution, we carry on the discussion only for the regular solution.

The regular solution (at $r = 0$) of eq.(2-11) is expressed as

$$\Phi(x) = x^{\gamma+1/2} e^{-x/2} F(\gamma+1+iy, 2\gamma+1; x) . \qquad (2\text{-}14)$$

Substituting eq.(2-14) into eq.(2-10), we obtain

$$\varphi_1 = x^{-1/2}\Phi(x)$$

$$= x^{\gamma} e^{-x/2} F(\gamma+1+iy,\ 2\gamma+1;\ x)$$

$$= i^{\gamma}(2pr)^{\gamma}\ e^{-ipr} F(\gamma+1+iy,\ 2\gamma+1;\ 2ipr)\ .$$

We set

$$\varphi_1 = N(\gamma+iy)e^{i\eta}\ (2pr)^{\gamma}\ e^{-ipr} F(\gamma+1+iy,\ 2\gamma+1;\ 2ipr)$$

$$= N(\gamma+iy)e^{i\eta}\ (2p)^{\gamma}\phi(r) \tag{2-15}$$

where N is a real normalization factor, and

$$\phi(r) = r^{\gamma} e^{-ipr} F(\gamma+1+iy,\ 2\gamma+1;\ 2ipr)\ . \tag{2-16}$$

The phase η is determined so that φ_2 in the eq.(2-5) is indeed φ_1^*. From eq.(2-15), we have

$$\varphi_2 = \varphi_1^* = N(\gamma-iy)e^{-i\eta}(2p)^{\gamma}\phi^*(r)\ . \tag{2-17}$$

Using eqs.(2-5) and (2-17), we obtain

$$\frac{1}{2ip} N(\gamma+iy)\ e^{i\eta}(2p)^{\gamma}\ \frac{d\phi}{dr}$$

$$= \left(\frac{1}{2} + \frac{\alpha ZW}{2p^2 r}\right) N(\gamma+iy)\ e^{i\eta}(2p)^{\gamma}\phi$$

$$-\left(\frac{\kappa}{2ipr} - \frac{\alpha Z}{2p^2 r}\right) N(\gamma-iy)\ e^{-i\eta}(2p)^{\gamma}\phi^*\ .$$

This can be reduced to a simpler form

$$\frac{d\phi}{dr} = ip\left(1 + \frac{y}{pr}\right)\phi - ip\left(\frac{\kappa}{ipr} - \frac{\alpha Z}{p^2 r}\right)\frac{\gamma - iy}{\gamma + iy}e^{-2i\eta}\phi^* \;.$$

From this equation $e^{-2i\eta}$ can be expressed as

$$e^{-2i\eta} = -\frac{\gamma + iy}{\gamma - iy}\,\frac{r}{\kappa - i\frac{y}{W}}\left[\frac{1}{\phi^*}\frac{d\phi}{dr} - ip\left(1 + \frac{y}{pr}\right)\frac{\phi}{\phi^*}\right] . \qquad (2\text{-}18)$$

Terms inside of the square brackets on the right hand side can be evaluated in the following way. We start with the defining equation (2-16) for ϕ

$$\phi(r) = r^{\gamma}\, e^{-ipr}\, F(\gamma+1+iy,\ 2\gamma+1;\ 2ipr) ,$$

hence

$$\phi^*(r) = r^{\gamma}\, e^{ipr}\, F(\gamma+1-iy,\ 2\gamma+1;\ -2ipr) .$$

We need to know dF/dr which appears in $d\phi/dr$. Using the relations for the confluent hypergeometric function

$$\frac{d}{dr}F(a,\ b;\ z) = \frac{a}{b}F(a+1,\ b+1;\ z) \quad {}^{13)}$$

and

$$zF(a+1,\ b+1;\ z) = b\left[F(a+1,\ b;\ z) - F(a,\ b;\ z)\right] ,$$

we can calculate dF/dr

$$\frac{d}{dr}F(\gamma+1+iy,\ 2\gamma+1;\ 2ipr)$$

$$= 2ip\frac{\gamma+1+iy}{2\gamma+1} F(\gamma+1+iy+1, 2\gamma+1+1; 2ipr)$$

$$= \frac{\gamma+1+iy}{r}\Big[F(\gamma+iy+2, 2\gamma+1; 2ipr) - F(\gamma+iy+1, 2\gamma+1; 2ipr)\Big] .$$

We are now ready for the evaluation of $d\phi/dr$.

$$\frac{d\phi}{dr} = \frac{d}{dr}\Big[r^{\gamma} e^{-ipr} F(\gamma+iy+1, 2\gamma+1; 2ipr)\Big]$$

$$= r^{\gamma} e^{-ipr} \Big[\frac{\gamma}{r} - ip - \frac{\gamma+iy+1}{r} F(\gamma+iy+1, 2\gamma+1; 2ipr)$$

$$+ \frac{\gamma+iy+1}{r} F(\gamma+iy+2, 2\gamma+1; 2ipr)\Big] .$$

We can therefore calculate the right hand side of eq.(2-18) making use of Kummer's relation: [13]

$$e^{-\frac{1}{2}z} F(a, b; z) = e^{\frac{1}{2}z} F(b-a, b; -z)$$

$$\frac{1}{\phi^*}\frac{d\phi}{dr} - ip(1 + \frac{y}{pr}) \frac{\phi}{\phi^*}$$

$$= -\left(\frac{1+2iy}{r} + 2ip\right) \frac{F(\gamma-iy, 2\gamma+1; -2ipr)}{F(\gamma+1-iy, 2\gamma+1; -2ipr)}$$

$$+ \frac{\gamma+iy+1}{r} \frac{F(\gamma-iy-1, 2\gamma+1; -2ipr)}{F(\gamma+1-iy, 2\gamma+1; -2ipr)}$$

$$= -\left(\frac{1+2iy}{r} + 2ip\right) \frac{F(a, b; z)}{F(a+1, b; z)} + \frac{\gamma+iy+1}{r} \frac{F(a-1, b; z)}{F(a+1, b; z)}$$

where we have defined a, b and z as

$$a = \gamma - iy, \quad b = 2\gamma+1, \quad z = -2ipr .$$

As the final step, we need several recurrence relations between the confluent hypergeometric functions. For the present purpose we use

$$aF(a+1, b; z) + (b-2a-z) F(a, b; z)$$
$$+ (a-b) F(a-1, b; z) = 0 .$$

We, then, readily obtain

$$\frac{1}{\phi^*}\frac{d\phi}{dr} - ip\left(1 + \frac{y}{pr}\right)\frac{\phi}{\phi^*} = \frac{\gamma - iy}{r} . \qquad (2\text{-}19)$$

Substituting eq.(2-19) into eq.(2-18), we have

$$e^{-2i\eta} = -\frac{\gamma + iy}{\kappa - \frac{iy}{W}} .$$

or

$$e^{2i\eta} = -\frac{\kappa - i\frac{y}{W}}{\gamma + iy} . \qquad (2\text{-}20)$$

This is the condition which η must satisfy in order that $F^c(r)$ and $G^c(r)$ be real functions. The left hand side of eq.(2-20) may be written as

$$e^{2i\eta} = \frac{1+i\tan\eta}{1-i\tan\eta}$$

so that we have

$$-\frac{\kappa - iy/W}{\gamma + iy} = \frac{1+i\tan\eta}{1-i\tan\eta} .$$

From this, we have

$$\kappa - \frac{y}{W}\tan\eta + \gamma - y\cdot\tan\eta + i\left(\gamma\tan\eta + y - \kappa\tan\eta - \frac{y}{W}\right) = 0 \quad .$$

Therefore, from the real part being zero, we have

$$\tan\eta = \frac{\kappa + \gamma}{(W+1)\frac{\alpha Z}{p}} \quad . \tag{2-21a}$$

Similarly, from the imaginary part, we have

$$\tan\eta = \frac{\alpha Z}{p}\,\frac{W-1}{\kappa - \gamma} \quad . \tag{2-21b}$$

Since both eqs.(2-21a) and (2-21b) must be satisfied simultaneously, we again have the condition

$$\gamma^2 = \kappa^2 - (\alpha Z)^2 \quad . \tag{2-9}$$

For the convenience, we here define $\bar{\eta}$ which satisfies the equation

$$\tan\bar{\eta} = \frac{\alpha Z}{p}\,\frac{W-1}{\kappa + \gamma} \quad . \tag{2-21c}$$

This equation can be obtained from eq.(2-21b) by replacing γ with $-\gamma$, so that $\bar{\eta}$ comes into question when the irregular solutions are considered. Using eq.(2-21b) and (2-21c), we obtain

$$\frac{\cos^2\bar{\eta}}{\cos^2\eta} = \frac{1+\left(\frac{\alpha Z}{p}\,\frac{W-1}{\kappa-\gamma}\right)^2}{1+\left(\frac{\alpha Z}{p}\,\frac{W-1}{\kappa+\gamma}\right)^2} = \frac{\kappa+\gamma}{\kappa-\gamma}\,\frac{W\kappa-\gamma}{W\kappa+\gamma} \quad . \tag{2-21d}$$

For the radial functions, we can now write, from eqs.(2-1), (2-2), (2-15) and (2-19)

$$F^c(r) = i(W-1)^{\frac{1}{2}} N(2pr)^{\gamma} \Big[(\gamma+iy)e^{-ipr+i\eta} F(\gamma+1+iy,\ 2\gamma+1;\ 2ipr) - c.c. \Big] \tag{2-22a}$$

$$G^c(r) = (W+1)^{\frac{1}{2}} N(2pr)^{\gamma} \Big[(\gamma+iy)e^{-ipr+i\eta} F(\gamma+1+iy,\ 2\gamma+1;\ 2ipr) + c.c. \Big] \tag{2-22b}$$

with $\quad e^{2i\eta} = - \dfrac{\kappa - i\frac{y}{W}}{\gamma + iy}$.

The factor $e^{i\eta}$ in eqs.(2-22a) and (2-22b) causes an ambiguity in sign of $F^c(r)$ and $G^c(r)$, but their ratio is unambiguous. To examine the asymptotic behaviour of the electron radial wave functions (2-22a) and (2-22b), we first consider the hypergeometric function at infinity. The asymptotic formulas for F(a, b; z) are [14]

$$F(a,\ b;\ z) \sim \begin{cases} \dfrac{\Gamma(b)}{\Gamma(a)}\, e^{z}\, z^{a-b} & (\mathrm{Re}\ \ z > 0) \\[2ex] \dfrac{\Gamma(b)}{\Gamma(b-a)}\, (-z)^{-a} & (\mathrm{Re}\ \ z < 0) \end{cases}$$

The relevant part in our case is

$$F(a,\ b;\ z) \sim \frac{\Gamma(b)}{\Gamma(a)}\ e^{z}\ z^{a-b} \tag{2-23}$$

Substituting eq.(2-23) into eqs.(2-22a) and (2-22b), we can write the asymptotic radial wave functions as

$$F^c(r) \longrightarrow i(W-1)^{\frac{1}{2}} N\Gamma(2\gamma+1)(2pr)^{\gamma} \times\left[\frac{(\gamma+iy)e^{ipr+i\eta}}{\Gamma(\gamma+1+iy)} (2ipr)^{iy-\gamma} - c.c.\right] \tag{2-24}$$

$$G^c(r) \longrightarrow (W+1)^{\frac{1}{2}} N\Gamma(2\gamma+1)(2pr)^{\gamma} \times\left[\frac{(\gamma+iy)e^{ipr+i\eta}}{\Gamma(\gamma+1+iy)} (2ipr)^{iy-\gamma} + c.c.\right] \tag{2-25}$$

Eqs.(2-24) and (2-25) can be reduced to somewhat familiar forms by using the identities

$$i^{-\gamma} = (e^{\frac{1}{2}i\pi})^{-\gamma} = e^{-\frac{1}{2}i\pi\gamma} ,$$

$$(2ipr)^{iy} = i^{iy}(2pr)^{iy} = (e^{\frac{1}{2}i\pi})^{iy}\, e^{\ln(2pr)^{iy}}$$
$$= e^{-\frac{1}{2}\pi y}\, e^{iy\ln(2pr)} ;$$

$$\frac{1}{\Gamma(\gamma+iy)} = \frac{e^{-i\arg\Gamma(\gamma+iy)}}{|\Gamma(\gamma+iy)|} ,$$

and $\Gamma(\gamma+iy+1) = (\gamma+iy)\Gamma(\gamma+iy)$.

We can now write the asymptotic forms of F^c and G^c as

$$F^c \xrightarrow{r\to\infty} i(W-1)^{\frac{1}{2}}\, N \frac{\Gamma(2\gamma+1)}{|\Gamma(\gamma+iy)|}\, e^{-\frac{1}{2}\pi y}$$

$$\times\left[e^{-i\arg\Gamma(\gamma+iy)}\, e^{iy\ln(2pr)}\, e^{-\frac{1}{2}i\pi\gamma}\, e^{i(pr+\eta)} - c.c.\right]$$

$$= -2\, (W-1)^{\frac{1}{2}}\, N \frac{\Gamma(2\gamma+1)}{|\Gamma(\gamma+iy)|}\, e^{-\frac{1}{2}\pi y}$$

$$\sin\left\{pr + y\ln(2pr) - \arg\Gamma(\gamma+iy) - \frac{\pi\gamma}{2} + \eta\right\}$$

$$= -A(W-1)^{\frac{1}{2}} \sin(pr + y\ln(2pr) + \delta_\kappa) \tag{2-26}$$

and

$$G^c \xrightarrow{r\to\infty} A(W+1)^{\frac{1}{2}} \cos(pr + y\ln(2pr) + \delta_\kappa) \tag{2-27}$$

where
$$A = 2N \frac{\Gamma(2\gamma+1)}{|\Gamma(\gamma+iy)|}\, e^{-\frac{1}{2}\pi y}\ , \tag{2-28}$$

$$\delta_\kappa = -\arg\Gamma(\gamma+iy) + \eta_\kappa - \frac{\pi\gamma}{2}\ . \tag{2-29}$$

We notice from eqs.(2-26) and (2-27) that F^c and G^c have a term ln(2pr) which is characteristic for the Coulomb potential. This can be true for any central potential that behaves as r^{-1} in the limit $r \longrightarrow \infty$. We must now determine the normalization constant A, or N. We normalize the wave function so that there is one particle in a very large sphere of radius R. From eqs.(2-26) and (2-27) the normalization in the sphere requires that

$$\int \Psi^* \Psi d\vec{r} = \int_0^R r^2 \, (f^2 + g^2) dr$$

$$= A^2 WR = 1 \quad . \qquad (2\text{-}30)$$

Here we have replaced the rapidly oscillating $\cos^2$ or $\sin^2$ by its mean value $\frac{1}{2}$. If the normalization is to one particle within a sphere of the unit radius, we have from eqs.(2-26) and (2-24)

$$A = \left(\frac{1}{W}\right)^{\frac{1}{2}} \qquad (2\text{-}31)$$

and

$$N = \frac{1}{2} \, \frac{|\Gamma(\gamma + iy)|}{\Gamma(2\gamma+1)} \, e^{\frac{1}{2}\pi y} \left(\frac{1}{W}\right)^{\frac{1}{2}} \quad . \qquad (2\text{-}32)$$

Using eq.(2-31) with eqs.(2-26) and (2-27), we thus can write the asymptotic radial wave functions normalized to one particle per unit sphere, in the following form.

$$F^c = -\sqrt{\frac{W-1}{W}} \, \sin\{pr + y\ln(2pr) + \delta_\kappa\} \qquad (2\text{-}33)$$

$$G^c = \sqrt{\frac{W+1}{W}} \, \cos\{pr + y\ln(2pr) + \delta_\kappa\} \qquad (2\text{-}34)$$

Substituing eq.(2-32) into eqs.(2-22a) and (2-22b) we note the radial wave function has the form

$$\begin{Bmatrix} F^c \\ G^c \end{Bmatrix} = \left(\frac{1 \mp W}{W}\right)^{\frac{1}{2}} (2pr)^{\gamma} \cdot e^{\frac{1}{2}\pi y} \frac{|(\gamma+iy)|}{(2\gamma+1)}$$

$$\times \frac{1}{2} \left\{ e^{-i(pr-\eta)} (\gamma+iy) \, F\,(\gamma+1+iy,\ 2\gamma+1;\ 2ipr) \mp c.c. \right\} . \tag{2-35a}$$

For the sake of completeness, we shall define $\overline{F}^c(r)$ and $\overline{G}^c(r)$, the irregular solutions for the Coulomb field, and these can be obtained, as already discussed, by changing γ to $-\gamma$ in F^c and G^c, respectively. We may write

$$\begin{Bmatrix} \overline{F}^c(r) \\ \overline{G}^c(r) \end{Bmatrix} = \left(\frac{1 \mp W}{W}\right)^{\frac{1}{2}} (2pr)^{-\gamma} \, e^{\frac{1}{2}\pi y} \frac{|\Gamma(-\gamma+iy)|}{\Gamma(-2\gamma+1)}$$

$$\times \frac{1}{2} \left\{ e^{-i(pr-\overline{\eta})} (-\gamma+iy) \, F\,(-\gamma+1+iy,\ -2\gamma+1;\ 2ipr) \mp c.c. \right\} . \tag{2-35b}$$

where $\overline{\eta}$ satisfies

$$\tan \overline{\eta} = \frac{\alpha Z}{p} \frac{W-1}{\kappa+\gamma} .$$

Eq. (2-35) is often written as

$$\begin{Bmatrix} F^c \\ G^c \end{Bmatrix} = \mp\left(\frac{W \mp 1}{W}\right)^{\frac{1}{2}} (2pr)^{\gamma} \, e^{\frac{1}{2}\pi y} \frac{|\Gamma(\gamma+iy)|}{\Gamma(2\gamma+1)} \begin{Bmatrix} \mathrm{Im}(\Lambda) \\ \mathrm{Re}(\Lambda) \end{Bmatrix} \tag{2-36}$$

where

$$\Lambda = e^{-ipr+i\eta} (\gamma+iy) \, {}_1F_1\,(\gamma+1+iy,\ 2\gamma+1;\ 2ipr)$$

and the phase η satisfies

$$e^{2i\eta} = \frac{\kappa - i\frac{\alpha Z}{p}}{\gamma + iy}$$

As is seen, the only factor in F^c and G^c which depends on the sign of κ is $e^{2i\eta}$. A function $\mathfrak{F}_0(W,Z)$ defined by

$$\mathfrak{F}_0(W,Z) = 4(2p\rho)^{2(\gamma_1 - 1)}\, e^{\pi y} \left\{ \frac{|\Gamma(\gamma_1 + iy)|}{\Gamma(2\gamma_1 + 1)} \right\}^2 \tag{2-37}$$

is called the Fermi function. The general solution is a linear combination of the regular and irregular solutions. Thus we can write

$$\left.\begin{aligned} F_\kappa(r) &= B\, F_\kappa^c(r) + C\, \overline{F}_\kappa^c(r) \\ G_\kappa(r) &= B\, G_\kappa^c(r) + C\, \overline{G}_\kappa^c(r) \end{aligned}\right\} \quad \text{for} \quad r > \rho \, .$$

The normalization condition for f_κ and g_κ corresponding to one particle per unit sphere yields

$$B^2 + C^2 + 2BC\cos(\delta_\kappa - \overline{\delta}_\kappa) = 1 \tag{2-38}$$

where $\delta_\kappa - \overline{\delta}_\kappa$ is the asymptotic phase difference between regular and irregular solutions.

So far we have discussed only for the Coulomb case. It remains to be seen how the Coulomb phase shifts δ_κ are altered by the assumption of a finite nuclear size. We already have seen a form of the potential due to the spherically symmetric charge distribution. (See equation(1-3)).

The asymptotic behavior of the radial wave function can be written, similar to those of the Coulomb potential,

$$F_\kappa = rf_\kappa \longrightarrow -\sqrt{\frac{W-1}{W}}\,\sin(pr + y\ln(2pr) + \Delta_\kappa) \tag{2-39}$$

$$G_\kappa = rg_\kappa \longrightarrow \sqrt{\frac{W+1}{W}}\,\cos(pr + y\ln(2pr) + \Delta_\kappa)\ . \tag{2-40}$$

Here Δ_κ still remains undetermined. Δ_κ is the phase shift that tends to δ_κ when a nucleus is reduced to a point nucleus. In the following let us denote the "inside" solutions of the Dirac equation as $f^{(i)}$ and $g^{(i)}$. As is in the Coulomb case, we define

$$F^{(i)}_\kappa = r\,f^{(i)}_\kappa \qquad \text{for} \quad r \leq \rho\ . \tag{2-41a}$$

$$G^{(i)}_\kappa = r\,g^{(i)}_\kappa \tag{2-41b}$$

At the nuclear radius ρ, "inside" and "outside" radial wave functions must be smootly connected. This requires, suppressing the index κ,

$$A\,F^{(i)}(\rho) = B\,F^c(\rho) + C\,\overline{F}^c(\rho) \tag{2-42a}$$

$$A\,G^{(i)}(\rho) = B\,G^c(\rho) + C\,\overline{G}^c(\rho) \tag{2-42b}$$

From eqs.(2-42a) and (2-42b), we obtain

$$A = \left(\frac{F^c/G^c - \overline{F}^c/\overline{G}^c}{F^{(i)}/G^{(i)} - \overline{F}^c/\overline{G}^c}\right)\frac{G^c}{G^{(i)}}\ B \tag{2-43a}$$

$$C = \left(\frac{F^c/G^c - F^{(i)}/G^{(i)}}{F^{(i)}/G^{(i)} - \bar{F}^c/\bar{G}^c} \right) \frac{G^c}{\bar{G}^c} B = HB \tag{2-43b}$$

where H is defined as

$$H = \frac{F^c/G^c - F^{(i)}/G^{(i)}}{F^{(i)}/G^{(i)} - \bar{F}^c/\bar{G}^c} \frac{G^c}{\bar{G}^c} \quad . \tag{2-44}$$

From the normalization condition, eq.(2-38), together with eq.(2-43), we get

$$B = \left[1 + H^2 + 2H \cos(\delta_\kappa - \bar{\delta}_\kappa) \right]^{-\frac{1}{2}} \tag{2-45}$$

In Bhalla-Rose' representation [4], phase shifts, exclusive of the logarithmic term, are given by

$$\tan \Delta_\kappa = \frac{-a_0 + a_1 \tan\eta_\kappa + H_\kappa \left(\frac{\cos\bar{\eta}}{\cos\eta}\right)_\kappa \left[a_2 + a_3 \tan\bar{\eta}_\kappa\right]}{a_1 + a_0 \tan\eta_\kappa + H_\kappa \left(\frac{\cos\bar{\eta}}{\cos\eta}\right)_\kappa \left[a_3 - a_2 \tan\bar{\eta}_\kappa\right]} \tag{2-46}$$

where

$$a_0 = \sin\tfrac{1}{2}\pi\gamma_\kappa \,\mathrm{Re}\, \frac{\Gamma(\gamma_\kappa + iy)}{|\Gamma(\gamma_\kappa + iy)|} + \cos\tfrac{1}{2}\pi\gamma_\kappa \,\mathrm{Im}\, \frac{\Gamma(\gamma_\kappa + iy)}{|\Gamma(\gamma_\kappa + iy)|}$$

$$a_1 = \cos\tfrac{1}{2}\pi\gamma_\kappa \,\mathrm{Re}\, \frac{\Gamma(\gamma_\kappa + iy)}{|\Gamma(\gamma_\kappa + iy)|} - \sin\tfrac{1}{2}\pi\gamma_\kappa \,\mathrm{Im}\, \frac{\Gamma(\gamma_\kappa + iy)}{|\Gamma(\gamma_\kappa + iy)|}$$

$$a_2 = \sin\tfrac{1}{2}\pi\gamma_\kappa \,\mathrm{Re}\, \frac{\Gamma(-\gamma_\kappa + iy)}{|\Gamma(-\gamma_\kappa + iy)|} - \cos\tfrac{1}{2}\pi\gamma_\kappa \,\mathrm{Im}\, \frac{\Gamma(-\gamma_\kappa + iy)}{|\Gamma(-\gamma_\kappa + iy)|}$$

$$a_3 = \cos\tfrac{1}{2}\pi\gamma_\kappa \,\mathrm{Re}\, \frac{\Gamma(-\gamma_\kappa + iy)}{|\Gamma(-\gamma_\kappa + iy)|} + \sin\tfrac{1}{2}\pi\gamma_\kappa \,\mathrm{Im}\, \frac{\Gamma(-\gamma_\kappa + iy)}{|\Gamma(-\gamma_\kappa + iy)|}$$

This can be simplified as

$$a_0 = \sin\tfrac{1}{2}\pi\gamma_\kappa \cos(\arg\Gamma(\gamma_\kappa + iy)) + \cos\tfrac{1}{2}\pi\gamma_\kappa \sin(\arg\Gamma(\gamma_\kappa + iy))$$

$$= \sin(\tfrac{1}{2}\pi\gamma_\kappa + \arg\Gamma(\gamma_\kappa + iy)) \quad . \tag{2-47a}$$

Similarly, we have

$$a_1 = \cos(\tfrac{1}{2}\pi\gamma_\kappa + \arg\Gamma(\gamma_\kappa + iy)) \tag{2-47b}$$

$$a_2 = \sin(\tfrac{1}{2}\pi\gamma_\kappa - \arg\Gamma(-\gamma_\kappa + iy)) \tag{2-47c}$$

$$a_3 = \cos(\tfrac{1}{2}\pi\gamma_\kappa - \arg\ (-\gamma_\kappa + iy)) \tag{2-47d}$$

Substituting eqs.(2-47) into eq.(2-46), we have

$$\tan\Delta_\kappa = \frac{\sin(\eta - \Theta) + H_\kappa\{\sin(\bar{\eta} + \Theta')\}}{\cos(\eta - \Theta) + H_\kappa\{\cos(\bar{\eta} + \Theta')\}}$$

where

$$\Theta = \tfrac{1}{2}\pi\gamma_\kappa + \arg\Gamma(\gamma_\kappa + iy)$$

$$\Theta' = \tfrac{1}{2}\pi\gamma_\kappa - \arg\Gamma(-\gamma_\kappa + iy) \quad .$$

Therefore

$$\tan\Delta_\kappa = \frac{\sin(\eta - \tfrac{1}{2}\pi\gamma_\kappa - \arg\Gamma(\gamma_\kappa + iy)) + \frac{C}{B}\sin(\bar{\eta} + \tfrac{1}{2}\pi\gamma_\kappa - \arg\Gamma(-\gamma_\kappa + iy))}{\cos(\eta - \tfrac{1}{2}\pi\gamma_\kappa - \arg\Gamma(\gamma_\kappa + iy)) + \frac{C}{B}\cos(\bar{\eta} + \tfrac{1}{2}\pi\gamma_\kappa - \arg\Gamma(-\gamma_\kappa + iy))}$$

$$= \frac{\sin\delta_\kappa + \frac{C}{B}\sin\bar{\delta}_\kappa}{\cos\delta_\kappa + \frac{C}{B}\cos\bar{\delta}_\kappa} \tag{2-48}$$

Note that when $H_\kappa = 0$ in eq.(2-46),

$$\tan\Delta_\kappa = \frac{-a_0 + a_1\tan\eta_\kappa}{a_1 + a_0\tan\eta_\kappa}$$

$$= \frac{-\sin\Theta + \cos\Theta \tan\eta_\kappa}{\cos\Theta + \sin\Theta \tan\eta_\kappa}$$

$$= \frac{\tan\eta_\kappa - \tan\Theta}{1 + \tan\Theta \tan\eta_\kappa} = \tan(\eta_\kappa - \Theta)$$

$$= \tan\left\{\eta_\kappa - \arg\Gamma(\gamma_\kappa + iy) - \tfrac{1}{2}\pi\gamma_\kappa\right\} = \tan\delta_\kappa$$

Thus, all informations concerning the nuclear size effect are contained in the terms multiplied by

$$H_\kappa\left(\frac{\cos\bar{\eta}}{\cos\eta}\right)_\kappa \text{ in eq. } \cancel{(2\text{-}46)}\ .$$

Now we can calculate the radial wave functions. From eqs.(2-41b), (1-12a) and (1-12b), it is found that at $r = \rho$

$$G_\kappa(\rho) = A\, G_\kappa^{(i)}(\rho) = \rho\, g_\kappa(\rho)$$

Thus

$$g_\kappa(\rho) = \frac{A}{\rho}\, G_\kappa^{(i)}(\rho)$$

$$= \frac{1}{\rho}\left(\frac{F^c/G^c - \bar{F}^c/\bar{G}^c}{F^{(i)}/G^{(i)} - \bar{F}^{(i)}/\bar{G}^{(i)}}\right)_\kappa G_\kappa^c\, B \qquad (2\text{-}49)$$

Similarly from eqs.(2-41a) and (2-41b), we have

$$f_\kappa(\rho) = \frac{F_\kappa^{(i)}(\rho)}{G_\kappa^{(i)}(\rho)}\, g_\kappa(\rho) \qquad (2\text{-}50)$$

To determine the radial wave functions $g_\kappa(\rho)$ and $f_\kappa(\rho)$, we have to know the ratio $F^{(i)}/G^{(i)}$. This ratio can be calcutated by the method of the power series expansion . It is found for

$\kappa < 0$ that

$$F^{(i)}(r) = r^{k+1}\sum_{n=0}^{\infty} A_n\, r^{2n} = r^{k+1}\sum_{n=0}^{\infty} a_n \quad , \tag{2-51}$$

$$G^{(i)}(r) = r^{k}\sum_{n=0}^{\infty} B_n\, r^{2n} = r^{k}\sum_{n=0}^{\infty} b_n \tag{2-52}$$

here $k = |\kappa|$.

Hence we find that at $r = \rho$

$$\frac{F^{(i)}(\rho)}{G^{(i)}(\rho)} = \rho\,\frac{\sum a_n}{\sum b_n} \tag{2-53}$$

where a_n and b_n satisfy the following recurrence relations.

$$a_0 = -\frac{W - 1 + 3\xi}{2k + 1}\, b_0 \tag{2-54a}$$

$$a_n = \frac{1}{2k + 2n + 1}\left[-(W - 1 + 3\xi)b_n + \xi b_{n-1} \right] \tag{2-54b}$$

$$b_n = \frac{\rho^2}{2n}\left[(W + 1 + 3\xi)a_{n-1} - \xi a_{n-2} \right] \tag{2-54c}$$

A proof of this result is given in Appendix D. For $\kappa > 0$, $F^{(i)}(\rho)/G^{(i)}(\rho)$ is obtained from eqs.(2-53) and (2-54) by changing the sign of W and Z and by interchanging $F^{(i)}$ and $G^{(i)}$.

CHAPTER 3

DIFFUSED NUCLEAR CHARGE DISTRIBUTION

In the preceeding section, the electron radial wave functions are obtained with the assumption of the uniform nuclear charge distribution. These radial wave functions have been tabulated by Bhalla and Rose[4]. It has been noted, however, that these wave functions do not approach the values for the point nucleus as the nuclear radius is made very small, and that these authors' phase convention for the positron wave functions is believed to have been explained inconsistently[7], and their values of the positron wave functions seem to behave very strangely at higher energy and at low atomic number[6].

To have a close look at these puzzling features of the wave functions, we introduce a diffused nuclear charge distribution, which reduces to the uniform charge distribution when a metric constant a approaches zero.

$$\rho_e(r) = \rho_e^0 \qquad (0 \le r \le \rho - t), \qquad (3\text{-}1a)$$

$$\rho_e(r) = \rho_e^0 \left[1 + \exp\left\{ -gd^{-1} 2gd((\rho - r)/a\rho) \right\} \right]^{-1}, \qquad (3\text{-}1b)$$

$$(\rho - t \le r \le \rho + t)$$

$$\rho_e(r) = 0 \qquad (\rho + t < r), \qquad (3\text{-}1c)$$

where gd is the Gudermannian function and gd^{-1} its inverse function. For the properties of the Gudermannian function, see Appendix E. The surface thickness is denoted by t which is related to the parameter a. The relation between t and a will be seen later. To see this charge distribution more clearly, we rewrite the eq.(3-1b) in the following way:

$$\left[1 + e^{-gd^{-1}2gd\frac{\rho - r}{a\rho}}\right]^{-1} = \frac{1}{2}\left(1 + \sinh\frac{\rho - r}{a\rho}\right).$$

The charge distribution can, thus, be written alternatively as

$$\rho_e(r) = \rho_e^0 \qquad (0 \leq r \leq \rho - t) \qquad (3\text{-}2a)$$

$$\rho_e(r) = \frac{\rho_e^0}{2}\left(1 + \sinh\frac{\rho - r}{a\rho}\right) \qquad (\rho - t \leq r \leq \rho + t) \qquad (3\text{-}2b)$$

$$\rho_e(r) = 0 \qquad (\rho + t < r) \qquad (3\text{-}2c)$$

Let us call each region correspondingly the "near origin", the "intermediate" and the "Coulomb" region in the equations (3-2). It may be worthwhile to compare our proposed charge distribution with Fermi distribution which has been extensively used for the electron scattering by a nucleus. Fermi distribution which is usually characterized by two parameters may be written as:

$$\rho_F(r) = \frac{\rho_F^0}{1 + e^{(r-c)/a_F}} = \frac{\rho_F^0}{2}\left(1 + \tanh\frac{c - r}{2a_F}\right) \qquad (3\text{-}3a)$$

where c is the radius at which the density is decreased by a factor of 2 below its central value and a_F is related to the surface thickness t_F, the distance over which the density falls from 90 percent of ρ_F^0 to 10 percent of ρ_F^0, through the equation

$$t_F = 4a_F \ln 3 \doteq 4.40\, a_F \quad .$$

The two parameter Fermi charge distribution may be improved to fit the scattering cross section data by introducing the third parameter w which can cause a hump or depression near the origin. The three parameter Fermi distribution[15] is written in the form

$$\rho_F(r) = (1 + w \frac{r^2}{c^2}) \frac{\rho_F^0}{1 + e^{(r-c)/a_F}} \qquad (3\text{-}3b)$$

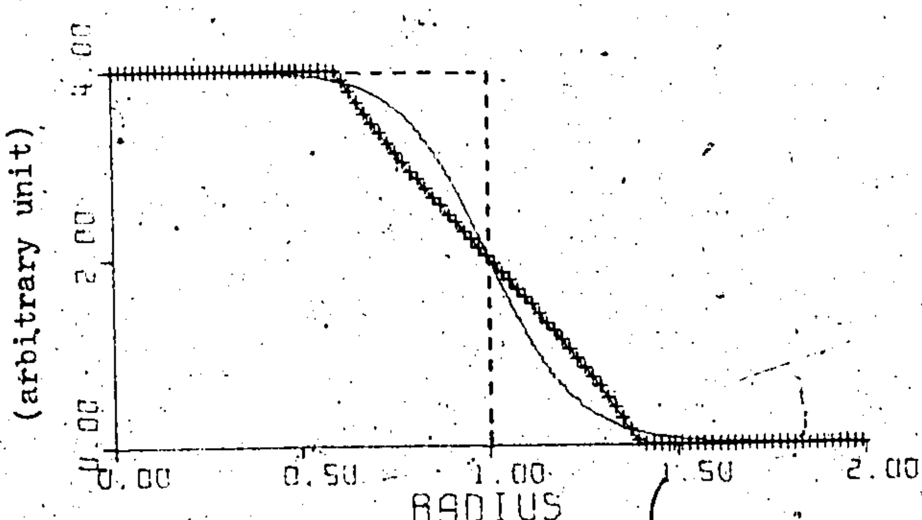

Fig. 3.1
Charge distribution by uniform (---), Fermi (——) and ours (+++).

Fig 3.1 shows the charge density distribution by uniform, two parameter Fermi and ours (eq.3-2). In analogy to the three parameter Fermi distribution, the sharp edge of ours may be made rounded by introducing an appropriate exponential factor multipled to eq.(3-1) as a smoothing factor. The word "appropriate" means that the smoothing factor does not alter the charge distribution appreciably.

In order to solve the Dirac equation, we have to know what the potential looks like. The potential associated with the diffused charge distribution can be obtained by the integral, as we have seen in Chapter 1,

$$V(r) = -4\pi Ze^2 \left[\frac{1}{r}\int_0^r \rho_e(r')r'^2dr' + \int_r^\infty \rho_e(r')r'dr'\right] \tag{3-4}$$

where $\rho_e(r)$ has been normalized by the condition

$$4\pi\int_0^\infty \rho_e(r')r'^2dr' = 1 \tag{3-5}$$

From the normalization (3-5) together with eqs.(3-2), it is found that the thickness t is related to the parameter $\underline{a}$ by the equation

$$t = a\rho\sinh^{-1}(1) \tag{3-6}$$

and that the charge density ρ_e^0 is also related to $\underline{a}$ by the expression

$$\rho_e^0 = \frac{1}{\frac{4\pi}{3}\rho^3}\,\frac{1}{1+3a^2(\sinh^{-1}(1)-\sqrt{2})^2} \tag{3-7}$$

For the derivation of eqs.(3-6) and (3-7), see Appendix F. Using eqs.(3-2) and (3-4) the potential due to the diffused charge distribution can be calculated analytically. It may be found that for the near origin region, $0 \leq r \leq \rho - t$,

$$V(r) = -\xi' \left[3 - \frac{r^2}{\rho^2} + 3a^2(\sinh^{-1}(1) - \sqrt{2}\,)^2 \right]$$
$$= -\xi' \left[3\left\{1 + a^2(\sqrt{2} - \sinh^{-1}(1))^2\right\} - \frac{r^2}{\rho^2} \right] \qquad (3\text{-}8a)$$

where
$$\xi' = \frac{\alpha Z}{2\rho}\left[1 + 3a^2(\sinh^{-1}(1) - \sqrt{2}\,)^2\right]^{-1} . \qquad (3\text{-}8b)$$

For the intermediate region, $\rho - t \leq r \leq \rho + t$, the form of the potential is rather complicated. The potential is essentially described in the form

$$V(r) = -\frac{1}{2}\,\xi'\left(3\,C_A - \frac{r^2}{\rho^2} + \frac{2\rho}{r}\,C_B\right) \qquad (3\text{-}8c)$$

where C_A and C_B are constants and both reduce to 1 when the thickness is taken to be zero. See Appendix G for a full derivation of eq.(3-8c). As is seen from eq.(3-8a), the potential tends to the one derived from the uniform charge distribution as a and, hence, t approach zero, since ξ' is also approaching ξ . If this is the case, we find that from eq.(3-8c)

$$V(r) = -\frac{\xi}{2}\left[3 - \frac{r^2}{\rho^2} + 2\,\frac{\rho}{r}\right] \tag{3-8d}$$

for the intermediate region. We again have the Coulomb potential for the region $\rho + t \leq r$

$$V(r) = -\frac{\alpha Z}{r} \; . \tag{3-8e}$$

The absolute value of the potential associated with the diffused charge distribution is shown in Fig 3.2.

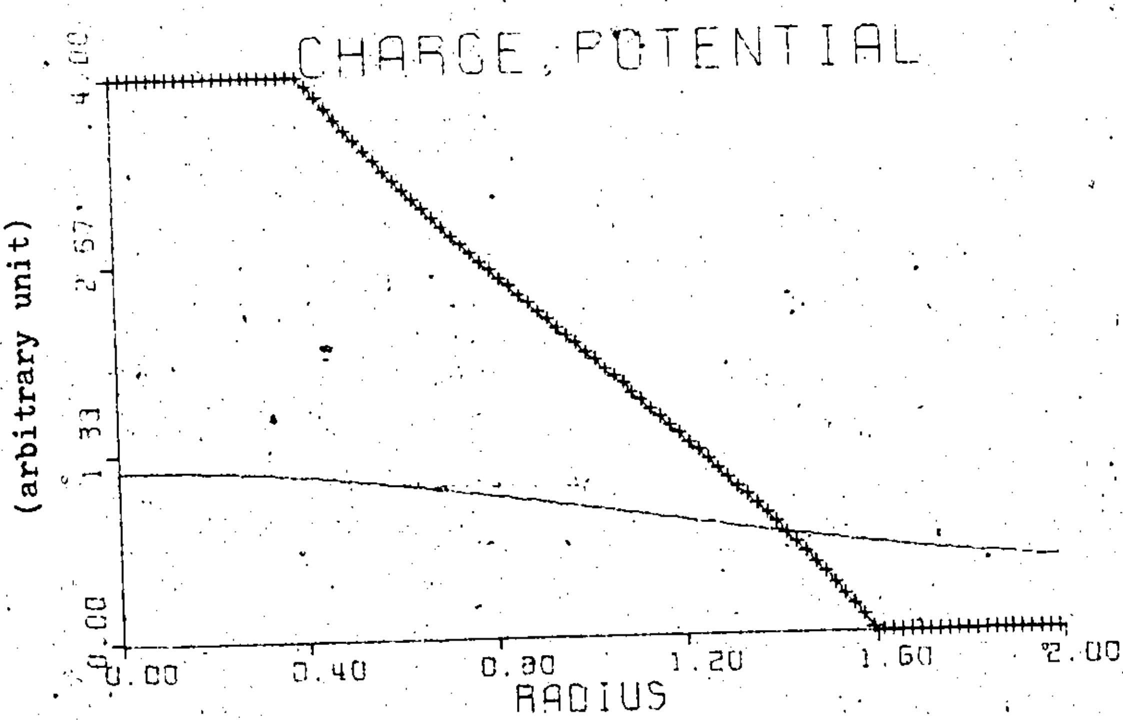

Fig 3.2 The diffused charge distribution (+++) and the potential (——).

Using this potential, we shall follow the prescription given in Chapter 1 in order to determine the radial wave functions. The procedure is, in principle, straightforward. Let $F^{(o)}$ and $G^{(o)}$ denote the radial wave function $f^{(o)}$ and $g^{(o)}$ multiplied by radial variable r, respectively, near the origin, and $F^{(i)}$ and $G^{(i)}$ in the intermediate region. See Fig. 3.3. As in the Coulomb case, the wave functions should be connected smoothly at each boundary so that at $r = \rho_1$, $F^{(o)}$ is equal to, in general, a

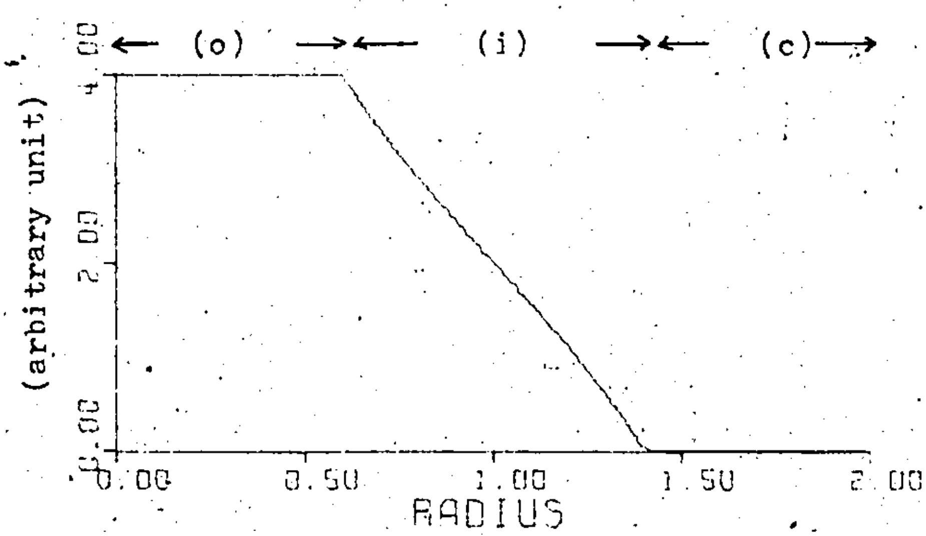

Fig 3.3 Three different regions of charge distribution, denoted by (o) the near origin, by (i) the intermediate and by (c) the Coulomb.

linear conbination of $F^{(i)^*}$ and $\overline{F}^{(i)}$, and at $r = \rho_2$, $F^{(i)}$ is equal to, in turn, a linear combination of F^c and $\overline{F}^c$. We may write

at $r = \rho_1$

$$A_0 F^{(o)}(\rho_1) = B_0 F^{(i)}(\rho_1) + C_0 \overline{F}^{(i)}(\rho_1) = \mathbb{F}^{(i)}, \qquad (3\text{-}9a)$$

$$A_0 G^{(o)}(\rho_1) = B_0 G^{(i)}(\rho_1) + C_0 \overline{G}^{(i)}(\rho_1) = \mathbb{G}^{(i)}, \qquad (3\text{-}9b)$$

and at $r = \rho_2$

$$A\, \mathbb{F}^{(i)}(\rho_2) = B\, F^c(\rho_2) + C\, \overline{F}^c(\rho_2), \qquad (3\text{-}10a)$$

$$A\, \mathbb{G}^{(i)}(\rho_2) = B\, G^c(\rho_2) + C\, \overline{G}^c(\rho_2). \qquad (3\text{-}10b)$$

In the eqs.(3-9a) and (3-9b), we can choose A_0 being 1 and $B_0 / A_0 = \widetilde{B}_0$ and $C_0 / A_0 = \widetilde{C}_0$ so that we have

$$\begin{pmatrix} F^{(i)} & \overline{F}^{(i)} \\ G^{(i)} & \overline{G}^{(i)} \end{pmatrix}_{\rho_1} \begin{pmatrix} \widetilde{B}_0 \\ \widetilde{C}_0 \end{pmatrix} = \begin{pmatrix} F^{(o)} \\ G^{(o)} \end{pmatrix}_{\rho_1}^{**} \qquad (3\text{-}11)$$

Thus

$$\widetilde{B}_0 = \left(\frac{F^{(o)}\overline{G}^{(i)} - G^{(o)}\overline{F}^{(i)}}{F^{(i)}\overline{G}^{(i)} - G^{(i)}\overline{F}^{(i)}} \right)_{\rho_1}$$

---

* $F^{(i)}$ should not be confused with the one used in Chapter 2. $F^{(i)}$ used in Chapter 2 rather corresponds to $F^{(o)}$ in here, so as $G^{(i)}$.

**A subscript ρ_1 is put aside to stress that each wave function is evaluated at $r = \rho_1$.

$$=\left(\frac{F^{(o)}/G^{(o)} - \bar{F}^{(i)}/\bar{G}^{(i)}}{F^{(i)}/G^{(i)} - \bar{F}^{(i)}/\bar{G}^{(i)}}\right)_{\rho_1} \left(\frac{G^{(o)}}{G^{(i)}}\right)_{\rho_1} , \tag{3-12a}$$

$$\tilde{C}_0 =\left(\frac{F^{(i)}/G^{(i)} - F^{(o)}/G^{(o)}}{F^{(i)}/G^{(i)} - \bar{F}^{(i)}/\bar{G}^{(i)}}\right)_{\rho_1} \left(\frac{G^{(o)}}{\bar{G}^{(i)}}\right)_{\rho_1} . \tag{3-12b}$$

Now, at $r = \rho_2$, in the eqs.(3-10a) and (3-10b), $\mathbb{F}^{(i)}$ itself is generally a linear combination of $F^{(i)}$ and $\bar{F}^{(i)}$. We may therefore write

$$A\,\mathbb{F}^{(i)}(\rho_2) = A\,(B_0F^{(i)}(\rho_2) + C_0\bar{F}^{(i)}(\rho_2)),$$

$$A\,\mathbb{G}^{(i)}(\rho_2) = A\,(B_0G^{(i)}(\rho_2) + C_0\bar{G}^{(i)}(\rho_2)),$$

so that

$$\left(\frac{\mathbb{F}^{(i)}}{\mathbb{G}^{(i)}}\right)_{\rho_2} = \left(\frac{B_0F^{(i)} + C_0\bar{F}^{(i)}}{B_0G^{(i)} + C_0\bar{G}^{(i)}}\right)_{\rho_2} = \frac{\tilde{B}_0F^{(i)}(\rho_2) + \tilde{C}_0\bar{F}^{(i)}(\rho_2)}{\tilde{B}_0G^{(i)}(\rho_2) + \tilde{C}_0\bar{G}^{(i)}(\rho_2)}$$

$$= \frac{B_0^{\prime}F^{(i)}(\rho_2)/G^{(i)}(\rho_1) + C_0^{\prime}\bar{F}^{(i)}(\rho_2)/\bar{G}^{(i)}(\rho_1)}{B_0^{\prime}G^{(i)}(\rho_2)/G^{(i)}(\rho_1) + C_0^{\prime}\bar{G}^{(i)}(\rho_2)/\bar{G}^{(i)}(\rho_1)} \tag{3-13}$$

where

$$B_0^{\prime} = (F^{(o)}/G^{(o)})_{\rho_1} - (\bar{F}^{(i)}/\bar{G}^{(i)})_{\rho_1} , \tag{3-14a}$$

and

$$C_0^{\prime} = (F^{(i)}/G^{(i)})_{\rho_1} - (F^{(o)}/G^{(o)})_{\rho_1} . \tag{3-14b}$$

We can easily show from eqs.(3-12) and (3-14) that

$$\frac{\widetilde{C}_0}{\widetilde{B}_0} = \frac{C_0'}{B_0'} \left(\frac{G^{(i)}}{\overline{G}^{(i)}} \right)_{\rho_1} \qquad (3\text{-}15)$$

We observe the following: eqs.(3-10a) and (3-10b) have the same form as eqs.(2-42a) and (2-42b) in Chapter 2, respectively, but the former two equations are expressed in terms of the linear combination of the regular and irregular solutions and the latter two are not. We now rewrite eq.(3-13) in the following form

$$\left(\frac{\mathbb{F}^{(i)}}{\mathbb{G}^{(i)}} \right)_{\rho_2} = \frac{1 + \dfrac{C_0'}{B_0'} \dfrac{\overline{F}^{(i)}(\rho_2)}{\overline{F}^{(i)}(\rho_1)} \left\backslash \dfrac{F^{(i)}(\rho_1)}{F^{(i)}(\rho_2)} \dfrac{\overline{F}^{(i)}(\rho_1)}{\overline{G}^{(i)}(\rho_1)} \dfrac{G^{(i)}(\rho_1)}{F^{(i)}(\rho_1)} \right.}{1 + \dfrac{C_0'}{B_0'} \dfrac{\overline{G}^{(i)}(\rho_2)}{\overline{G}^{(i)}(\rho_1)} \dfrac{G^{(i)}(\rho_1)}{G^{(i)}(\rho_2)}} \quad \frac{F^{(i)}(\rho_2)}{G^{(i)}(\rho_2)} \qquad (3\text{-}16)$$

In order to determine the radial wave functions f and g, we may make use of the same method as was described in Chapter 2, especially eqs.(2-49) and (2-50) but $F^{(i)}/G^{(i)}$ should be replaced by $\mathbb{F}^{(i)} / \mathbb{G}^{(i)}$ of eq.(3-16).

We now calculate the ratio of the wave functions by the method of the power series expansion. For the "near origin" region, the power series solutions are already known. We can use eqs.(2-53) and (2-54) to evaluate the ratio of $F^{(o)}/G^{(o)}$ only by making changes

(BR) (ours)

$$\xi \longrightarrow \xi'$$

$$\mathfrak{z} \longrightarrow \mathfrak{z}\left\{1 + a^2(\sqrt{2} - \sinh^{-1}(1))^2\right\}$$

$\rho \longrightarrow \rho - t$, i.e., we evaluate the ratio at $r = \rho - t$.

Thus we have

$$\left.\frac{F^{(o)}}{G^{(o)}}\right|_{r=\rho-t} = \frac{F_0}{G_0}(\rho-t)\frac{\Sigma a_n}{\Sigma b_n} \tag{3-17}$$

and the recurrence relations for a_n and b_n are

$$a_n = \frac{1}{2k+2n+1}\left[-(W-1+3\xi')b_n + \xi' \frac{(\rho-t)^2}{\rho} b_{n-1}\right] \tag{3-18a}$$

$$b_n = \frac{(\rho-t)^2}{2n}\left[(W+1+3\xi')a_{n-1} - \xi' \frac{(\rho-t)^2}{2} a_{n-2}\right]. \tag{3-18b}$$

For the intermediate region, $F^{(i)}$ and $G^{(i)}$ are expressed in the form

$$F^{(i)}(r) = F_0\, r^{\gamma} \Sigma A_n r^n = F_0\, r^{\gamma} \Sigma a_n \tag{3-19a}$$

$$G^{(i)}(r) = G_0\, r^{\gamma} \Sigma B_n r^n = G_0\, r^{\gamma} \Sigma b_n \tag{3-19b}$$

where

$$\frac{F_0}{G_0} = \frac{\gamma + \kappa}{\xi' \rho\, C_B} \tag{3-20}$$

and

$$\gamma^2 = \kappa^2 - (\xi' \rho\, C_B)^2 . \tag{3-21a}$$

Note that γ in the eq.(3-21a) is different from those defined in Chapter 2, through the equation (2-9), $\gamma^2 = \kappa^2 - (\alpha Z)^2$. In the eq.(3-21a), when the thickness is vanishingly small, we have

$$\gamma^2 = \kappa^2 - \left(\frac{\alpha Z}{2} \right)^2 . \tag{3-21b}$$

Comparing eq.(3-21b) with eq.(2-9), it is found that the nuclear charge Ze in eq.(3-21b) is effectively reduced to half as much as used to be. We also note that if $F^{(i)}$, for example, is evaluated at the outer radius $r = \rho + t$, a_n should be expressed as

$$a_n = A_n(\rho + t)^n .$$

Similarly at the inner radius $r = \rho - t$, $a_n = A_n(\rho - t)^n$ must be used. The recurrence relations for a_n and b_n are now expressed as follows: at $r = \rho + t$

$$a_n = - \frac{\xi' C_B \rho(\rho+t)}{n(n+2\gamma)} \left[(W + 1 + \frac{3\xi'}{2} C_A^{out})a_{n-1} - \frac{\xi'(\rho+t)^2}{2\rho^2} a_{n-3} + (1 + \frac{n}{\gamma+\kappa}) \left\{ (W - 1 + \frac{3\xi'}{2} C_A^{out})b_{n-1} - \frac{\xi'(\rho+t)^2}{2\rho^2} b_{n-3} \right\} \right] , \tag{3-22a}$$

$$b_n = - \frac{\xi' C_B \rho(\rho+t)}{n(n+2\gamma)} \left[(W - 1 + \frac{3\xi'}{2} C_A^{out})b_{n-1} - \frac{\xi'(\rho+t)^2}{2\rho^2} b_{n-3} + (1 - \frac{n}{\kappa-\gamma}) \left\{ (W + 1 + \frac{3\xi'}{2} C_A^{out})a_{n-1} - \frac{\xi'(\rho+t)^2}{2\rho^2} a_{n-3} \right\} \right] \tag{3-22b}$$

and at $r = \rho - t$, we can obtain the recurrence relations from eqs.(3-22a) and (3-22b) by simply replacing C_A^{out} with C_A^{in} and $(\rho+t)$ with $(\rho-t)$. A more detailed derivation of equations (3-19) to (3-22) can be found in Appendices G and H. We are now ready to calculate the radial wave functions f and g.

CHAPTER 4

BEHABIOUR OF LEPTONIC PROPERTIES IN BETA DECAYS

In the preceding chapters, we have only discussed the electron radial wave functions without mentioning why these wave functions are needed in the framework of the theory of beta-decay. In other words we have not indicated so far that how the electron radial wave functions are related to the observable quantities which can be determined by the experiment. In order to show this, it seems inevitable to discuss the classical theory of beta-decay. The classical theory of beta-decay which does not take into account the parity nonconsevation may be good enough for our present purpose and can be still used to interprot the results of many experiments, such as beta-decay lifetimes, electron-neutrino angular correlation and beta-spectrum. In developing a theory of beta-decay, we have to apply the tools of the quantum field theory, since, in a beta transition, particles are created or annihilated. Although we use the results obtained from the field theory, we do not go further into this subject. One may refer to the references 16) or 17) for the account on the field theory.

The basic beta processes are

$$n \longrightarrow p + \beta^- + \bar{\nu}, \quad \beta^- \text{decay}; \qquad (4\text{-}1a)$$

$$p \longrightarrow n + \beta^+ + \nu, \quad \beta^+ \text{decay;} \tag{4-1b}$$

$$p + e^- \longrightarrow n + \nu, \quad \text{Electron capture (EC),} \tag{4-1c}$$

where e^- is a bound electron. The process (4-1a) is energetically possible for a free neutron to undergo. The processes (4-1b) and (4-1c) are energetically forbidden in free space unless there is an extra supply of energy which may be present in the field of other nucleons inside of a nucleus. For a more complex nucleus, the decay is of the form

$$Z^A \longrightarrow (Z + 1)^A + \beta^- + \bar{\nu} .$$

The electron and the antineutrino carry off practically all the energy released in the decay, neglecting a small recoil energy transferred to the nucleus. The energy release is given by

$$W_0 = ({}_Z M^A - {}_{Z+1}M^A)\ c^2$$

where M denotes the atomic mass of the nuclide involved.

The positron decay may be expressed as

$$Z^A \longrightarrow (Z - 1)^A + \beta^+ + \nu$$

so that the energy release W_0 is then written as

$$W_0 = ({}_Z M^A - {}_{Z-1}M^A - 2m_e)\ c^2 .$$

The electron capture process reduces the number of elementary charge by one unit, and at the same time removes one electron from one of the atomic shells, most often a K electron. The process, therefore, leaves the daughter atom excited, and a cascade of orbital electrons will, in general, follow the capture. The energy release is

$$W_0 = ({}_{Z}M^A - {}_{Z-1}M^A)\, c^2 - \varepsilon_e$$

where ε_e refers to the binding energy of the captured electron. We shall call "allowed transitions" the transitions in which both the electron (positron) and the antineutrino (neutrino) are emitted with zero orbital angular momentum, and we shall call "forbidden transitions" the relatively weaker ones in which either one or both particles are emitted with an orbital angular momentum different from zero. Hereafter we restrict our discussions to the case of the allowed beta decay. Since the both light particles have the intinsic spin $\frac{1}{2}$, these spins can combine to a total intrinsic spin j_t=0 (singlet state) or 1 (triplet state). The transitions for which j_t=0 are called allowed Fermi transitions; those for which j_t=1 are called allowed Gamow-Teller transitions. As a result of the anglar momentum consevation the value j_t determines selection rules for the change in the angular momentum of the nucleus, as given by

$$\left| I_i - I_f \right| \leq j_t \leq I_i + I_f \quad .$$

Setting $j_t = 0$ or $j_t = 1$ we obtain the selection rules for allowed transitions. Thus for Fermi transitions, there can be no change in angular momentum of the nucleus, whereas for Gamow-Teller transitions, the change in nuclear angular momentum can be either one unit or zero, but zero to zero transitions are excluded. Furthermore, since the both leptons are emitted in S states ($\ell = 0$) which have even parity, there can not be any change in the parity of the nuclear wave functions.

The interaction Hamiltonian density for a general beta decay can be expressed in the form

$$\mathcal{H} = \sum_i \left[g_i (\bar{\psi}_p O_i \psi_n)(\bar{\psi}_e O_i \psi_\nu) + \text{h.c.} \right] , \qquad (4\text{-}2)$$

where $\quad i = S, V, T, A, P.$

The meaning of the operator O_i is summarized in Table 4.1.

Table 4.1 Form of interaction

Operator O_i	Number of independent matrices	Relativistic transformation properties of $\bar{\psi}_a O_i \psi_b$
1	1	Scalar (S)
γ_μ	4	Vector (V)
$\gamma_\mu \gamma_\nu \ (\mu \neq \nu)$	6	Tensor (T)
$\gamma_5 \gamma_\mu$	4	Axial vector (A)
$\gamma_5 = \gamma_1 \gamma_2 \gamma_3 \gamma_4$	1	Pseudo scalar (P)

We followed the notation used in the reference 17). The g's are coupling constants which specify the strength of the beta decay interaction. In the quantum field theory ψ_p, ψ_n, ψ_e, and ψ_ν are operators describing the annihilation of a proton, neutron, electron and neutrino, respectively, and at the same time the creation of the corresponding antiparticles. A nucleus consists of A nucleons and it is equally possible for each nucleon to undergo a beta transition. Introducing the isotopic spin operator τ_k^+ (τ_k^-) which carries out the transformation of the k-th neutron (proton) into a proton (neutron), we make the following replacement in eq.(4-2),

$$\bar{\psi}_p O_i \psi_n \longrightarrow \bar{\psi}_p O_i \left(\sum_{k=1}^{A} \tau_k^+\right) \psi_n \qquad (4-3)$$

We have included in eq.(4-2) all the possible interactions. A correct combination of the interaction Hamiltonian densities in beta decay, i.e., a combination out of the five possible relativistic invariants, can be determined by the experiment.

Let us now consider the way in which the available energy is shared between the beta particle and the neutrino, i.e., the shape of the beta spectrum. The probability for emission of an electron with energy W per unit time can be written as

$$N(W)dW = \frac{2\pi}{\hbar} \left| \mathcal{H}_{fi} \right|^2 \frac{dN_F}{dW_0} \qquad (4-4)$$

where $\mathcal{H}_{fi}$ is the transition matrix element from an initial state (i) to a final state (f) and dN_F/dW is the density of the final states available to the system per unit range of the total energy W_0. Since the recoil energy of the daughter nucleus is very small, W_0 is equal to the sum of the energies carried off by the electron and the antineutrino for β^- decay

$$W_0 = W_e + W_{\bar{\nu}} \tag{4-5}$$

The neutrino has practically no interaction with matter and its wave function is simply a plane wave

$$\psi_\nu = (2\pi)^{-3/2} u_\nu e^{i(\vec{k}_\nu \cdot \vec{r})} = (2\pi)^{-3/2} u_\nu e^{-i(\vec{k}_{\bar{\nu}} \cdot \vec{r})} \tag{4-6a}$$

where $\vec{k}_\nu$ is the wave vector and u_ν is a spinor describing the polarization of the neutrino and $\vec{k}_{\bar{\nu}}$ is the wave vector of the emitted antineutrino.

The electron emitted from the nucleus, on the other hand, has an electric charge and the wave function will be distorted by the electromagnetic interaction with the nuclear charge (Coulomb distortion). For the following discussion we shall disregard this distortion, however, and shall assume for the moment that the electron wave function can also be approximated by a plane wave

$$\psi_e = (2\pi)^{-3/2} u_e e^{i(\vec{k}_e \cdot \vec{r})} \tag{4-6b}$$

The interaction density is then composed of terms of the form

$$\bar{\psi}_e O_i \psi_\nu = (2\pi)^{-3}\, \bar{u}_e O_i u_\nu e^{-i(\vec{k}_e+\vec{k}_{\bar{\nu}})\cdot\vec{r}}$$

$$= (2\pi)^{-3}\, \bar{u}_e O_i u_\nu \left[1 - i(\vec{k}_e+\vec{k}_{\bar{\nu}})\cdot\vec{r} - \frac{\{(\vec{k}_e+\vec{k}_{\bar{\nu}})\cdot\vec{r}\}^2}{2!} + \cdots \right] \qquad (4\text{-}6c)$$

In calculating the transition matrix element, we only have to integrate over the nuclear volume, since the nuclear wave functions are practically zero outside of the nucleus. Hence the lepton functions are needed for $0 \le r \le \rho$ only. For leptons with energies of about 1 MeV, we have $k \sim 5\times10^{-3}$ fermi^{-1} and if ρ is, say, about 10 fermi we have $(k_e + k_{\bar{\nu}})\rho \sim 1/10$. As the transitiion probability is proportional to the square of $\mathcal{H}_{fi}$, consecutive terms in an expansion will differ by a factor $(k_e + k_{\bar{\nu}})^2\rho^2$, i.e., by almost two orders of magnitude. Transitions generated by the first term of the expansion are called allowed and transitions associated with higher order terms are called forbidden. The statement on "allowed" and "forbidden" mentioned here is equivalent to what was mentioned earlier. Since, if only the first term of the series in eq.(4-6c) is kept, we are led to a nuclear matrix element, for instance,

$$\int 1 = \int dV \sum_k \bar{\psi}\tau_k\psi = \langle f | \sum_k \tau_\pm^k | i \rangle \,. \qquad (4\text{-}7)$$

This is the matrix element for the scalar interaction so that it can be different from zero only if the initial and the final states have the same total angular momentum I and parity. This implies that the orbital angular momentum of the leptons is zero. We expect therefore that for allowed transitions there is no correlation between the two lepton momenta and hence no electron-neutrino angular anisotropy. However, it has been shown in chapter 1 that the relativistic wave functions contain the orbital angular momentum $\ell=1$ besides $\ell=0$. The contribution of "small" components of the relativistic wave function is proportional to v/c (v is the velocity of the electron and c the velecity of light) and if v/c is not small compared to 1 such relativistic angular correlations have to be expected even for the allowed transitions. On the other hand, it is often a good approximation to treat, at least, the neucleons nonrelativistically. We justfy the application of nonrelativistic methods by noting that nucleons move with velocities of only $\sim$ 1/10 c in nucleus.

A more accurate expression [18] for the allowed beta spectrum including the Coulomb correction is give by

$$N(W)dW = \frac{g_V^2}{2\pi^3} \mathfrak{F}_0(\pm Z,W)pW(W_0-W)^2 L_0 \left[(\textstyle\int 1)^2 + \lambda^2 (\int \vec{\sigma})^2 \right] \tag{4-8}$$

where $\int \vec{\sigma} = \langle f \,|\sum_k \tau_+^k \vec{\sigma}|\, i \rangle$,

$$\lambda = - g_A/g_V \quad , \tag{4-9}$$

$$L_0 = (2p^2 \mathfrak{F}_0)^{-1}(g_{-1}^2 + f_1^2) \quad , \tag{4-10a}$$

and $\mathfrak{F}_0$ is the Fermi function defined earlier via eq.(2-37). The above spectral form is derived under the following assumptions: The electron radial wave functions f_1 and g_{-1}, which are originally in the integrand of the nuclear matrix element, can be taken outside of the integral and are replaced by the values evaluated ~~at~~ the nuclear radius ρ, even though they might have certain radial dependence. As is seen from eq.(4-8), the Coulomb correction apperars in the two parts, the Fermi function and the symbol L_0. The Coulomb correction enhances the probability of electron emission and decreases the probability of positron emission especially at low energies.

So far the discussion has been made only for the allowed beta decay which invloves a spin change of 0 or 1 and no change of parity. If this is not the case, the beta decay will still occur with a relatively smaller transition probability. We will then need the higher- order powers of $(\vec{k}_e + \vec{k}_\nu)\cdot\vec{r}$ in the expansion of eq.(4-6c), correspondingly called "first forbidden", "second forbidden" and so on. The beta spectra for the forbidden transitions have been calculated[19,20]. As is L_0, the following symbols are customarily used for representing the combinations of the electron wave functions in these formulas.

$$N_0 = (2p^2\mathfrak{F}_0)^{-1}\rho^{-1}\left[f_{-1}g_{-1} - f_1g_1\right] \tag{4-11a}$$

$$M_0 = (2p^2\mathfrak{F}_0)^{-1}\rho^{-2}\left[g_1^2 + f_{-1}^2\right] \tag{4-12a}$$

$$L_1 = (2p^2\mathfrak{F}_0)^{-1}\rho^{-2}\left[g_{-2}^2 + f_2^2\right] \tag{4-13a}$$

$$L_{12} = (2p^2\mathfrak{F}_0)^{-1}\rho^{-1}\left[g_{-1}f_2\cos(\Delta_{-1} - \Delta_2) - f_1g_{-2}\cos(\Delta_1 - \Delta_{-2})\right]^* \tag{4-14a}$$

$$N_{12} = (2p^2\mathfrak{F}_0)^{-1}\rho^{-2}\left[f_{-1}f_2\cos(\Delta_{-1} - \Delta_2) + g_1g_{-2}\cos(\Delta_1 - \Delta_{-2})\right] \tag{4-15a}$$

When αZ is much smaller than unity, eqs.(4-10a) through (4-15a) can be approximated by the simpler forms indicated by arrows;

$$L_0 = (1/2)(1+\gamma) - (5/3)\alpha Z\rho W - (1/3)(\alpha Z\rho/W) - (1/3)p^2\rho^2 + ---$$

$$\longrightarrow 1 \quad , \tag{4-10b}$$

$$N_0 = -(p^2/3W)\gamma - \xi + (2/3)(\alpha Z)^2W - (p^2/9W)(\alpha Z)^2$$

$$+ (11/18)p^2\alpha Z\rho^2 + ---$$

$$\longrightarrow -(p^2/3W) - \xi \, . \tag{4-11b}$$

$$M_0 \longrightarrow p^2/9 + (2/3)\xi p^2/W + \xi^2 \tag{4-12b}$$

---

* The symboles L_{12} and N_{12} will appear respectively in the β-ν and β-γ angular correlation formulas.

$$L_1 \longrightarrow p^2/9 \quad , \tag{4-13b}$$

$$L_{12} \longrightarrow - p^2/ 3W \tag{4-14b}$$

$$N_{12} \longrightarrow p^2/9 + \xi(p^2/3W) \tag{4-15b}$$

These expressions are called the Konopinski-Uhlenbeck approximation[19]. Even if αZ is small, it is preferred to include several higher order terms when the electron energy becomes higher (see eq.(4-11b) or (4-12b)). The deviation from the Konopinski-Uhlenbeck approximation is called the finite deBroglie wavelength effect.

In the next chapter we will discuss the validity of the BR method of evaluation of positron radial wave functions along with the combinations of the radial wave functions (L_0 to N_{12}), and also discuss the influence of the diffused charge distribution in beta decays.

CHAPTER 5

DISCUSSION AND CONCLUSION

We first discuss the BR method for the evaluation of the positron radial wave functions. As was mentioned several times in the preceding chapters, the positron radial wave functions of BR are believed to be in error[21]. The possible reasons of BR's failure have been pointed out by Huffaker and Laird[6] and by Bühring[7]. The former authors' argument is based on the phase convention of η which is introduced in order to make the wave functions real, while the latter on the Z-dependence of the sign convention of these phases. Although we follow the BR's prescription to obtain the radial wave functions, we adopt a different phase convention for η . The phases η and $\bar{\eta}$ satisfy the following expressions;

$$e^{2i\eta} = -\frac{\kappa - iy/W}{\gamma + iy} = -\frac{\gamma - iy}{\kappa + iy/W} , \tag{5-1}$$

$$e^{2i\bar{\eta}} = -\frac{\kappa - iy/W}{-\gamma + iy} = \frac{\gamma + iy}{\kappa + iy/W} . \tag{5-2}$$

From eqs.(5-1) and (5-2), we have

$$e^{2i(\eta - \bar{\eta})} = \frac{-\gamma + iy}{\gamma + iy} . \tag{5-3}$$

We now choose the phases η and $\bar{\eta}$ such that, for electron emission, the difference $(\eta-\bar{\eta})$ lies in the first quadrant and that, for positron emission, $(\eta-\bar{\eta})$ lies in the fourth quadrant. In this way, we have calculated radial wave functions $f_{\pm 1}$, $g_{\pm 1}$ and the ratio $f_{\pm 1}/g_{\pm 1}$ for nuclei (A,Z) = (56,-26) and (228,-90). The comparison between BR, Bühring[7] and ours is shown in Table 5.1, where ours(i) is obtained by the modified BR method and ours(ii) by method used in the electron scattering (see reference 29)). As is shown in Table 5.1, our results are in excellent agreement with Bühring's.

Assuming the uniform charge distribution of a nucleus, Huffaker and Laird[6] calculated the radial wave functions for the beta decays from ^{12}B and ^{12}N by using the approximate radial wave functions expanded in powers of (αZ) or (pr), of which first three terms being taken into account. For the comparison of their results with those of BR, they plot the quantities f_{+1}/c_- and g_{-1}/c_+, which are equal to each other in thier approximation, as functions of W, where

$$c_{\pm} = \left[(W\pm 1)W^{-1}p^2\mathfrak{F}_0(W,Z)\right]^{\frac{1}{2}}$$

and $\mathfrak{F}_0$ is the Fermi function. Their figure shows a marked discrepancy between their values and BR's values for positron emission. In order to see if our method reproduces the right trend of the wave functions of Huffaker and Laird, we have

Table 5.1 Positron radial wave functions evaluated at the nuclear surface

Z (A)	p	Ref.	f_{+1}	g_{+1}	f_{-1}	g_{-1}	f_{+1}/g_{+1}	f_{-1}/g_{-1}
-26 (56)	1.0	BR	.380777E0	-.253258E-1	.668674E-1	.897665E0	-.150352E2	.744904E-1
		Bühring	.380765E0	-.253248E-1	.668518E-1	.897453E0	-.150350E2	.744906E-1
		Ours(i)	.380760E0	-.253246E-1	.668510E-1	.897445E0	-.150352E2	.744904E-1
		(ii)	.380761E0	-.253247E-1	.668512E-1	.897447E0		
	6.0	BR	.429426E1	-.205273E0	.282462E0	.506402E1	-.209198E2	.557711E-1
		Bühring	.428518E1	-.204834E0	.281592E0	.504914E1	-.209203E2	.557703E-1
		Ours(i)	.428514E1	-.204837E0	.281594E0	.504911E1	-.209198E2	.557711E-1
		(ii)	.428516E1	-.204838E0	.281595E0	.504912E1		
-90 (228)	1.0	BR	.275233E0	-.712184E-1	.138052E0	.507674E0	-.386463E1	.271930E0
		Bühring	.274496E0	-.710278E-1	.136820E0	.503143E0	-.386463E1	.271931E0
		Ours(i)	.274497E0	-.710281E-1	.136820E0	.503146E0	-.386463E1	.271930E0
		(ii)	.274497E0	-.710280E-1	.136820E0	.503146E0		
	6.0	BR	.340049E1	-.766942E0	.942008E0	.394809E1	-.443383E1	.238598E0
		Bühring	.323908E1	-.730532E0	.883843E0	.370433E1	-.443386E1	.238597E0
		Ours(i)	.323907E1	-.703536E0	.883846E0	.370433E1	-.443383E1	.238598E0
		(ii)	.323908E1	-.730538E0	.883847E0	.370433E1		

calculated the same quantities f_{+1}/c_- and g_{-1}/c_+ for the uniformly charged nucleus retaining all higher order terms in the wave functions. The use has been made of eq.(2-36) where the series of the confluent hypergeometric functions are terminated when the absolute value of the n-th term of the expansion becomes less than 10^{-7}. To see how the wave functions vary with the charge distribution inside of a nucleus, we have also calculated f_{+1}/c_- and g_{-1}/c_+ for the weak coagulation limit[22) which will be explained later. The results are shown in Fig 5.1, indicating that our results exhibit the same behaviour as those of Huffaker and Laird. However our accurate numerical calculation shows that the values g_{-1}/c_+ are slightly larger than f_{+1}/c_- in all energy considered.

To summarize the first part of the discussion, the positron radial wave functions are found to be remarkably consistent with Bühring's and with Huffaker and Laird's, and also found to be in excellent agreement with those derived through the electron scattering analysis. We therefore conclude that our method of the evaluation of the radial wave functions is correct.

Secondly we have examined how the diffused nuclear charge distribution affects the radial wave functions. Using the potential for the diffused charge distribution developed in chapter 3, we have calculated the radial wave functions for various values of the thickness. It turns out

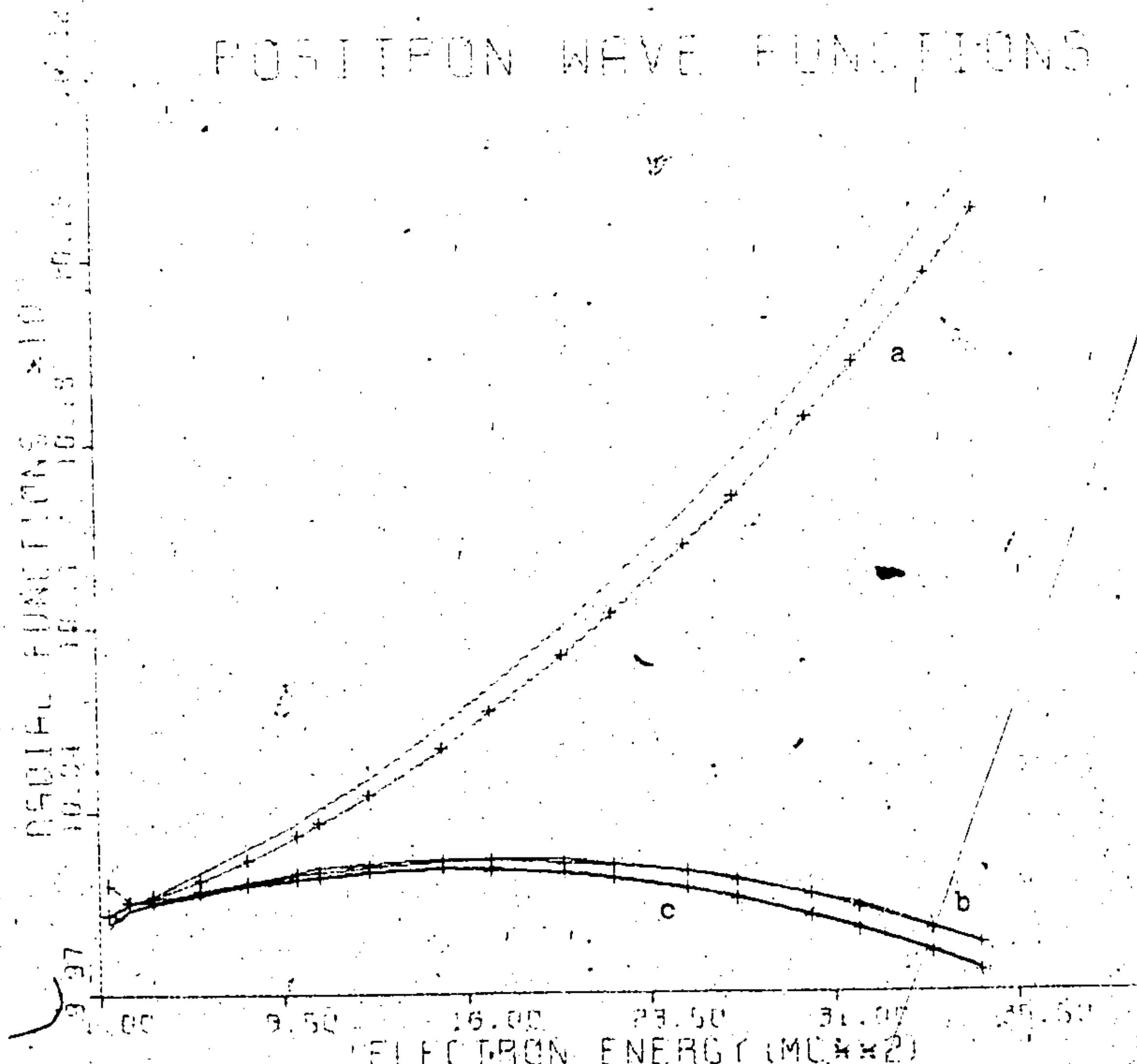

Fig 5.1 f_{+1}/c_- (with +) and g_{-1}/c_+ plotted for the decay $^{12}N \longrightarrow {}^{12}C$. Curve a,b,c designates BR method, ours and weak coagulation limit, respectively.

that the radial wave functions evaluated at the nuclear radius are the same as those of the uniform charge distribution as long as there is a finite central region of the uniform charge adjacent to the diffused surface. However, when the potential in the intermediate region is extended to the whole nuclear region, the radial wave functions change considerably for larger Z values. Let us call this limit "weak coagulation" limit. The form of the potential was already given by eq.(3-8d). As is expected from the potential form and from Fig 5.1 (together with the reference 6)), "weak coagulation" limit (WCL) may be considered as a finite size version of the point charge nucleus. Table 5.2 shows the values of the electron wave functions in different models for the beta decay processes $^{210}Bi \longrightarrow {}^{210}Po$ and $^{12}B \longrightarrow {}^{12}C$, where the thikness is taken to be 2.5 fermi for the diffuse charge model in both cases. This suggests that the ratio (F_K/G_K) at the inner boundary may be strongly dependent on Z. To see the situation more clearly, we have calculated combinations of these electron wave functions L_0, N_0, M_0, L_1, L_{12} and N_{12}, and the results are shown in Fig 5.2. The broken line shows the uniform charge distribution calculation and our diffused charge distribution, whereas the heavy solid line indicates the weak coagulation limit. The two other curves are the one calculated from Konopinski-Uhrenbeck formula (dash-dot line) and that of the point nucleus approximation (thin solid line), added for comparison. When the nuclear radius ρ is made smaller, together

Table 5.2 Electron radial wave functions evaluated at the nuclear surface

A (Z)	p	Model	f_{+1}	g_{+1}	f_{-1}	g_{-1}
210 (90)	1.0	Diffused	-.402835E1	-.109502E1	-.194014E1	.748917E1
		WCL	-.416292E1	-.146472E1	-.263919E1	.778067E1
		Uniform	-.402835E1	-.109502E1	-.194014E1	.748917E1
	6.0	Diffused	-.194666E2	-.593912E1	-.641826E1	.219652E2
		WCL	-.204310E2	-.795894E1	-.872388E1	.231534E2
		Uniform	-.194666E2	-.593912E1	-.641826E1	.219652E2
12 (6)	1.0	Diffused	-.598704E0	-.139171E-1	-.267010E-1	.144320E1
		WCL	-.598769E0	-.165431E-1	-.330327E-1	.144345E1
		Uniform	-.598704E0	-.139171E-1	-.267010E-1	.144320E1
	6.1	Diffused	-.588558E1	-.202065E0	-.205490E0	.694541E1
		WCL	-.588710E1	-.227965E0	-.236061E0	.694764E1
		Uniform	-.588558E1	-.202065E0	-.205490E0	.694541E1

with Z, it is observed that the last three curves become almost completely overlapping with each other. The broken line, however, does not follow the same pattern, for instance, L_0 is almost completely parallel to the one derived from the weak coagulation limit. On the other hand, if the nuclear radius is kept unchanged but Z is made smaller, the broken line moves toward the other curves. This fact represents the strong influence of the uniformly charged central region onto

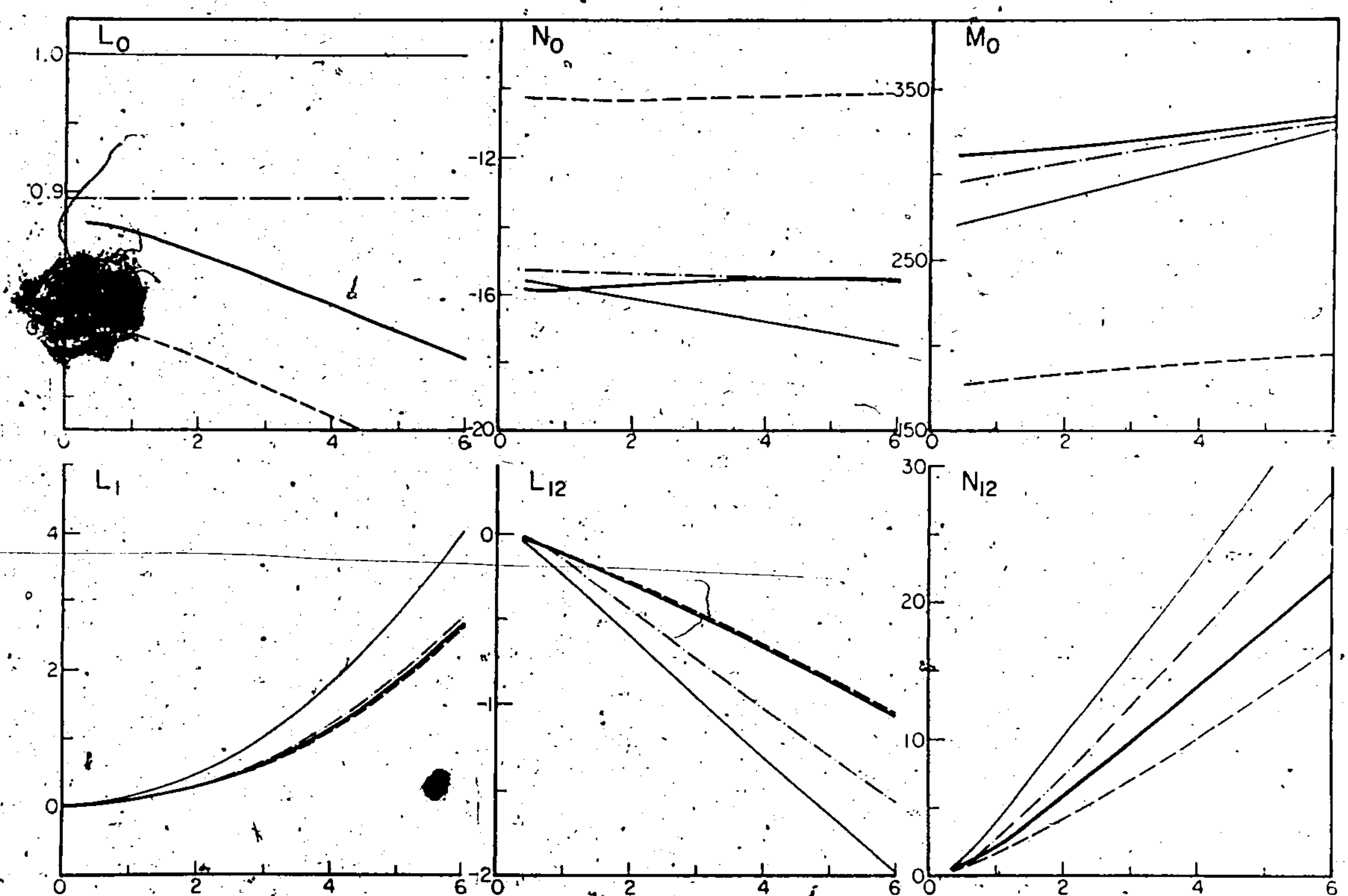
L0
1.0
0.9
0
2
4
6
N0
-12
-16
-20
0
2
4
6
M0
350
250
150
0
2
4
L1
4
2
0
0
2
4
6
L12
0
-1
-2
0
2
4
6
N12
30
20
10
0
0
2
4
6

Fig 5.2

the raitio (E_κ/G_κ) at the inner boundary.

The similar situation occurs for the positron decays except that these curves themselves change very much when the nuclear radius ρ and Z are made smaller. Fig 5.3 shows the

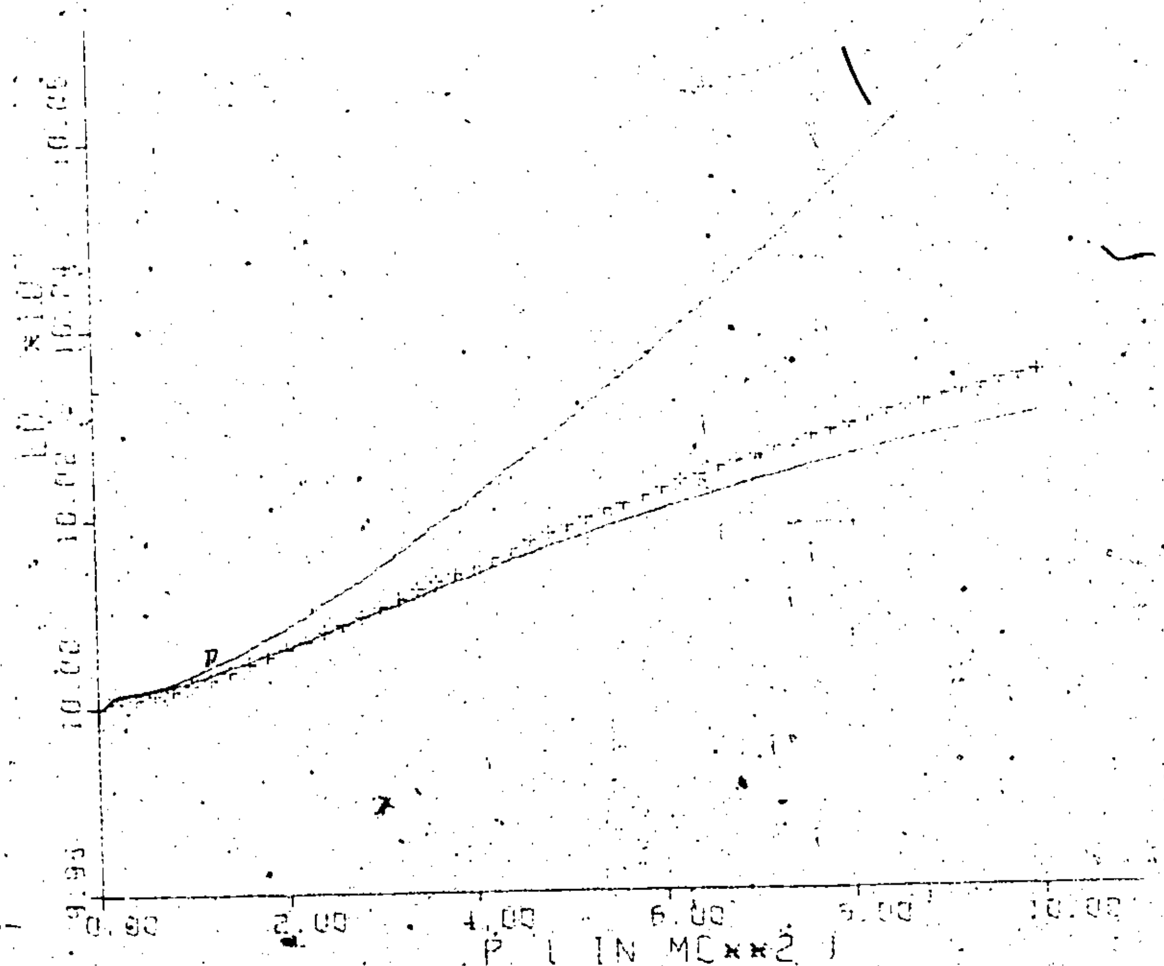

Fig 5.3 L_0 for decay $^{12}N \longrightarrow {}^{12}C$. The upper solid line indicates BR's calculation and the lower solid line indicates the weak coagulation limit. The diffused charge distribution calculation is indicated by (+++).

value L_0 for positron decay $^{12}N \longrightarrow {}^{12}C$ where BR's curve is also added for comparison. We note from Fig 5.3 that for increasing p BR method gives more diverging values away from the classical limit. We have also calculated L_0 and N_0 for positron decay $^{228}Pa \longrightarrow {}^{228}Th$ and the results are shown in Fig 5.4a and b.

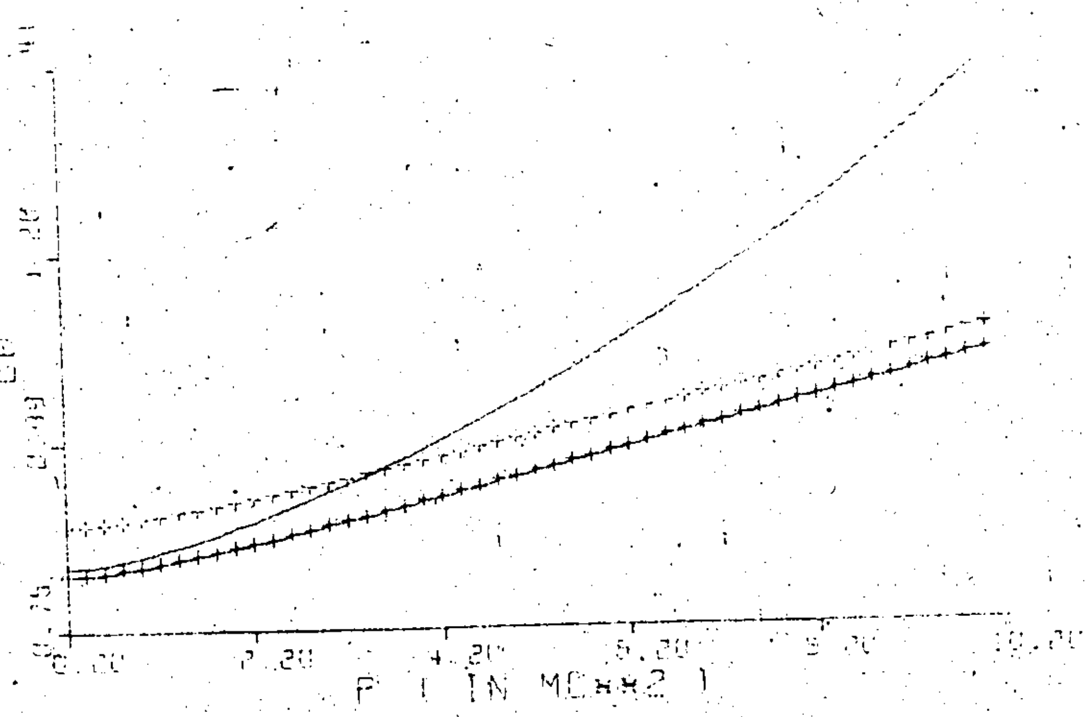

Fig 5.4a L_0 for decay $^{228}Pa \longrightarrow {}^{228}Th$. Values of BR, the diffused and WCL are indicated by the solid line, the solid line with (+++) and (+++), respectively.

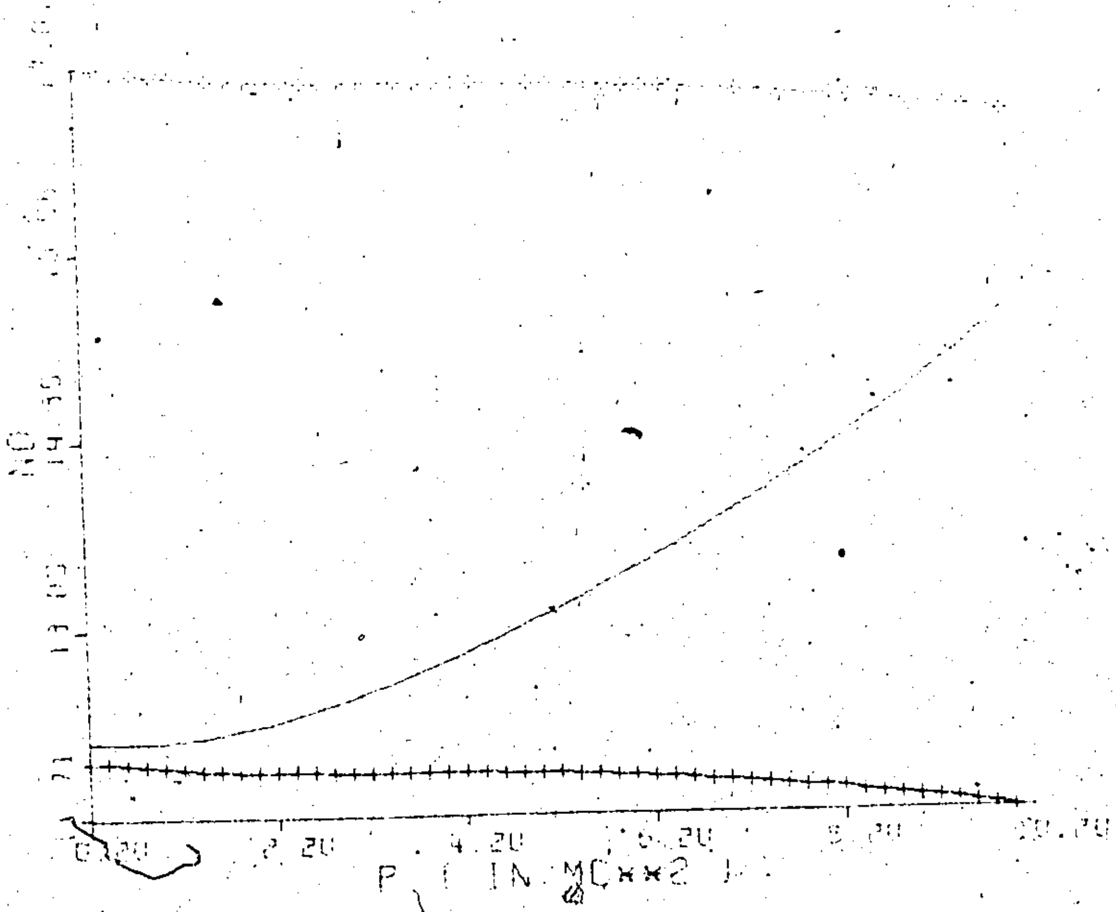

Fig 5.4b N_0 for decay $^{228}Pa \longrightarrow {}^{228}Th$.

See the caption of Fig 5.4a.

Finally there has been a controversy[23,24] about whether or not the detailed charge distribution has some effects in the analysis of beta decay and the more realistic type of distribution, like Fermi of Gaussian, has been investigated along this line. To see if the assumption of the uniform charge distribution inside of a nucleus remains valid or not, we have calculated the radial wave functions for the decay $^{88}Rb \longrightarrow {}^{88}Sr$ with the diffused and with the uniform charge distribution, both of which are so adjusted as to yield the same root mean square (r.m.s.) radius. (For the derivation of the r.m.s. radius of the diffused charge distribution, see Appendix I.) We take the r.m.s. radius to be 4.31 fermi which

Table 5.3 Radial wave functions associated with the different charge distributions as functions of momentum (in the relativistic units).

	Uniform		Diffused	
p	f_1	g_{-1}	f_1	g_{-1}
1.0	-.1145E1	.2615E1	-.1147E1	.2620E1
2.2	-.3053E1	.4641E1	-.3059E1	.4651E1
5.4	-.8171E1	.9740E1	-.8191E1	.9763E1
10.0	-.1518E2	.1670E2	-.1522E2	.1675E2

would be a reasonable radius from the electron scattering data (see also reference 34)). This corresponds, for example, to the nuclear radius $\rho = 1.25A^{1/3}$ fermi for the uniform charge, and $\rho = 1.20A^{1/3}$ fermi for the diffused charge distribution with the thickness of 2.0 fermi. The results are shown in Table 5.3. Since the percentage difference in the wave functions is about 0.3% and since transition probability is propotional to the square of the matrix element, the assumption of the uniform charge distribution may remain satisfactory in the analysis of beta decay at present.

PART II

HIGH ENERGY ELECTRON SCATTERING

CHAPTER 6

ELASTIC ELECTRON SCATTERING

The electromagnetic interaction of charged particles with nuclei is the most important source of information about the charge distribution in nuclei. The electron scattering experiment has been extensively carried out by Hofstadter et al[25]. For electrons of very low energy the scattering by a nucleus can be attributed to a point charge, but at energies above about 50 MeV the detailed behavior of the nuclear charge distribution has a substantial effect on the scattering process. This can be seen by considering the reduced de Broglie wave length of the electron. To detect finite size effects, it is necessary that the reduced de Broglie wave length should be comparable to the size of the nucleus under consideration. For this to be the case, the electron energy must be in the extreme relativistic region and it was realized by Acheson[26] that the rest mass energy of the electron may be neglected in comparison with the total electron energy. This assumption makes the Dirac equation conceivably simpler. It should be also mentioned that the interaction between the projectile and the target is a function of their relative positions, i.e., $\vec{r} = \vec{r}_1 - \vec{r}_2$, it is therefore convenient to use the center-of-mass system, since this yields an equation of motion for a single particle with coordinate $\vec{r}$ moving in the potential $V(r)$.

The relation between the cross sections in the laboratory and center-of-mass coordinate systems can be found in the references 27) and 28).

We first follow the formalism of Yennie[29] et.al's prescriptions. In their representation, the Dirac equation can be written as

$$(\vec{\alpha}\cdot\vec{p}c + \beta mc^2 + V)\psi = E\psi \; . \tag{6-1}$$

We choose the following representation

$$\vec{\alpha} = \begin{pmatrix} \vec{\sigma} & 0 \\ 0 & -\vec{\sigma} \end{pmatrix}, \qquad \beta = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \; . \tag{6-2}$$

The fourth rank matrices are expressed in terms of the second rank matrices. If we write ϕ and χ for two component wave functions, the Dirac equation (6-1) can be written

$$(\vec{\sigma}\cdot\vec{p}c + V - E)\phi = -mc^2\chi \; , \tag{6-3a}$$

$$(-\vec{\sigma}\cdot\vec{p}c + V - E)\chi = -mc^2\phi \; , \tag{6-3b}$$

and if the electron rest mass is neglected, we see that the equations become uncoupled,

$$(\vec{\sigma}\cdot\vec{p}c + V - E)\phi = 0 \; , \tag{6-4a}$$

$$(-\vec{\sigma}\cdot\vec{p}c + V - E)\chi = 0 \; . \tag{6-4b}$$

In the absence of a potential, we have

$$(\vec{\sigma}\cdot\vec{p} - E)\phi = 0 ,$$

$$(-\vec{\sigma}\cdot\vec{p} - E)\chi = 0 .$$

These equations are called the Weyl equations. The equations have plane wave solutions

$$\phi = u\, e^{i\vec{k}\cdot\vec{x}} ,$$

$$\chi = v\, e^{i\vec{k}\cdot\vec{x}}$$

where $\vec{\sigma}\cdot\vec{p}cu = Eu ,$ (6-5a)

$\vec{\sigma}\cdot\vec{p}cv = -Ev ,$ (6-5b)

and $\vec{p} = \hbar\vec{k} = \hbar k\hat{k}$, $\hat{k}$ is a unit vector in the direction of $\vec{p}$. For a given $E(>0)$ and $\vec{k}$, these two solutions correspond to the two different spin states of the electron. This can be easily seen by considering the helicity h defined by

$$h = \frac{\vec{\sigma}\cdot\vec{p}}{|\vec{p}|} = \pm 1 .$$

From eq.(6-5a) and (6-5b), we have

$\vec{\sigma}\cdot\hat{k}u = u ,$ (6-6a)

$\vec{\sigma}\cdot\hat{k}v = -v ,$ (6-6b)

so that the solutions u and v correspond to spin directions parallel and anti-parallel to the direction of the momentum, respectively. It may be interesting to note that at the high energy limit ($m=0$) the spin is always parallel or anti-parallel to the momentum, so that in a scattering process the spin direction is turned through the same angle as the momentum is. Let θ denote the polar angle and φ the azimuthal angle of the unit vector $\hat{k}$. We have then

$$\vec{\sigma}\cdot\hat{k} = \begin{pmatrix} \cos\theta & \sin\theta e^{-i\varphi} \\ \sin\theta e^{i\varphi} & -\cos\theta \end{pmatrix} . \qquad (6\text{-}7)$$

Substituting eq.(6-7) into eqs.(6-6a) and (6-6b), we have the spinor parts of the plane wave solutions corresponding to an electron moving in the direction (θ,φ) with spin direction (θ,φ) and $(\pi-\theta,\pi+\varphi)$ respectively, one for the parallel direction to the momentum and the other anti-parallel.

$$u = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} \cos\frac{\theta}{2} \\ \sin\frac{\theta}{2}\, e^{i\varphi} \end{pmatrix} , \qquad (6\text{-}8a)$$

$$v = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} -\sin\frac{\theta}{2}\, e^{-i\varphi} \\ \cos\frac{\theta}{2} \end{pmatrix} . \qquad (6\text{-}8b)$$

In other words, the spin functions which diagonalize $\vec{\sigma}\cdot\hat{k}$ with eigenvalue ± 1 have the forms shown in eqs.(6-8a) and (6-8b). Now the potential V is brought back into the Dirac equation which describes an electron scattered in the field V. Asymptotically the solution of this equation consists of a plane wave and a scattered spherical wave, whose radial dependence is of the form $r^{-1} \cdot \exp(ikr)$. Since the potential vanishes in the asymptotic region, the spinor part of the scattered wave must be identical with that for a free wave, although the amplitude $f(\theta,\varphi)$ of the wave scattered into any particular direction(θ,φ) will depend on the scattering process. We take $\theta=0$ for the incident beam, corresponding to the z-axis being parallel to the incident momentum. Thus the scattering states will have the asymptotic form

$$\phi \sim \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{ikz} + \begin{pmatrix} \cos\frac{\theta}{2} \\ \sin\frac{\theta}{2}\, e^{i\varphi} \end{pmatrix} f_1(\theta,\varphi)\, \frac{e^{ikr}}{r} \quad , \quad (6\text{-}9a)$$

$$\chi \sim \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{ikz} + \begin{pmatrix} -\sin\frac{\theta}{2}\, e^{-i\varphi} \\ \cos\frac{\theta}{2} \end{pmatrix} f_2(\theta,\varphi)\, \frac{e^{ikr}}{r} \quad . \quad (6\text{-}9b)$$

In the case of a spherically symmetric potential, the scattering cross-section

$$\frac{d\sigma}{d\Omega} = \left| f(\theta,\varphi) \right|^2 \qquad (6\text{-}10)$$

is the same for either spin direction so that we restrict our attention to the ϕ-solution.

We wish to obtain the solution of the equation (6-4a) whose asymptotic form is given by eq.(6-9a). We now decompose

ϕ into partial waves ϕ_{jm}, which are eigenstates of the total angular momentum and its z component. Thus

$$\phi = \sum_{j,m} a_{jm}\phi_{jm} \tag{6-11}$$

with

$$\vec{J}^2\phi_{jm} = j(j+1)\hbar^2\phi_{jm} \tag{6-12a}$$

$$J_z\phi_{jm} = m\hbar\phi_{jm} \tag{6-12b}$$

where

$$\vec{J} = \vec{L} + \vec{S} = \vec{r}\times\vec{p} + \frac{\hbar}{2}\vec{\sigma} \tag{6-12c}$$

$$J_z = xp_y - yp_x + \frac{\hbar}{2}\sigma_z \tag{6-12d}$$

since

$$J_z \begin{pmatrix}1\\0\end{pmatrix} e^{ikz} = \frac{1}{2}\hbar\begin{pmatrix}1\\0\end{pmatrix} e^{ikz}\ , \tag{6-12e}$$

the incident plane-wave in the z-direction has $m=\frac{1}{2}$ and we shall be concerned only with the states $\phi_{j,\frac{1}{2}}$. In terms of the spherical harmonics $Y_{\ell m}(\theta,\varphi)$ and the spin vectors

$$\alpha = \begin{pmatrix}1\\0\end{pmatrix}\ , \qquad \beta = \begin{pmatrix}0\\1\end{pmatrix}^{*}$$

the eigenfunctions for $J_z = \frac{1}{2}$ are given by

---

* Note that α and β must not be confused with the $\vec{\alpha}$ and β of the Dirac equation.

$$\chi_j = c_+ Y_{\ell 0}\alpha + c_- Y_{\ell 1}\beta \tag{6-13}$$

where c_+ and c_- are constants and will be determined from the fact that χ_j is to be an eigenfunctions of $\vec{J}^2$. In general, a scalar product $\vec{u}\cdot\vec{v}$ is written in terms of the spherical components as

$$\vec{u}\cdot\vec{v} = \tfrac{1}{2}(u_+v_- + u_-v_+) + u_0v_0$$

where $u_\pm = u_x \pm iu_y$, $u_0 = u_z$ and so on.

Now
$$\begin{aligned}\vec{J}^2 &= \vec{L}^2 + \vec{S}^2 + 2\vec{L}\cdot\vec{S}\\ &= \vec{L}^2 + \vec{S}^2 + L_-S_+ + L_+S_- + 2L_zS_z\end{aligned} \tag{6-14}$$

and it follows from the properties of the spherical harmonics and the spin vectors that

$$L_+Y_{\ell 0} = \hbar\sqrt{\ell(\ell+1)}\, Y_{\ell 1},\quad L_-Y_{\ell 1} = \hbar\sqrt{\ell(\ell+1)}\, Y_{\ell 0}$$

$$S_+\alpha = 0,\quad S_+\beta = \alpha\hbar,\quad S_-\alpha = \beta\hbar,\quad S_-\beta = 0.$$

We thus find that

$$\begin{aligned}2\vec{L}\cdot\vec{S}\,\chi_j &= (L_-S_+ + L_+S_- + 2L_zS_z)(c_+Y_{\ell 0}\alpha + c_-Y_{\ell 1}\beta)\\ &= \hbar^2 c_-\sqrt{\ell(\ell+1)}\, Y_{\ell 0}\alpha + \hbar^2(c_+\sqrt{\ell(\ell+1)} - c_-)\, Y_{\ell 1}\beta\end{aligned} \tag{6-15}$$

From eq.(6-14) we have

$$\vec{J}^2\chi_j = \left\{ l(l+1)\hbar^2 + \frac{3}{4}\hbar^2 \right\}\chi_j + 2\vec{L}\cdot\vec{S}\chi_j.$$

since χ_j is an eigenfunction of $\vec{J}^2$, we must have

$$2\vec{L}\cdot\vec{S}\chi_j = \lambda\hbar^2\chi_j = \lambda\hbar^2(c_+Y_{l0}\alpha + c_-Y_{l1}\beta) \tag{6-16}$$

where $\lambda = \vec{J}^2/\hbar^2 - l(l+1) - 3/4$ (6-17)

Comparing coefficients of $Y_{l0}\alpha$ and $Y_{l1}\beta$ in eqs.(6-15) and (6-16), the following equation can be obtained.

$$\lambda c_+ = c_-\sqrt{l(l+1)} \tag{6-18a}$$

$$(\lambda+1)c_- = c_+\sqrt{l(l+1)} \tag{6-18b}$$

Hence $(\lambda-l)(\lambda+l+1) = 0$

so that $\lambda = l$ or $-l-1$.

In the case of $\lambda = l$, using eq.(6-17), we have

$$\vec{J}^2 = \hbar^2(l+\tfrac{1}{2})(l+3/2)$$

and from eq.(6-18a) we find that

$$\frac{c_+}{c_-} = \sqrt{\frac{l+1}{l}}$$

Similarly we may find for $\lambda = -l-1$ that

$$\vec{J}^2 = \hbar^2(l-\tfrac{1}{2})(l+\tfrac{1}{2})$$

and $$\frac{c_+}{c_-} = -\sqrt{\frac{\ell}{\ell+1}} .$$

Corresponding to two values of λ, we thus have the following eigenfunctions and eigenvalues

$$\chi_j^1 = (\sqrt{\ell+1}\, Y_{\ell 0}\alpha + \sqrt{\ell}\, Y_{\ell 1}\beta)\frac{N}{\sqrt{2\ell+1}} , \quad j = \ell + \tfrac{1}{2} \qquad (6\text{-}19a)$$

$$\chi_j^2 = (\sqrt{\ell}\, Y_{\ell 0}\alpha - \sqrt{\ell+1}\, Y_{\ell 1}\beta)\frac{N}{\sqrt{2\ell+1}} , \quad j = \ell - \tfrac{1}{2} \qquad (6\text{-}19b)$$

where $\frac{N}{\sqrt{2\ell+1}}$ is an arbitrary normalization constant.

We rewrite eqs.(6-19) in terms of the Legendre polynomials using the relation

$$Y_{\ell m}(\theta,\varphi) = \sqrt{\frac{2\ell+1}{4\pi}\frac{(\ell-m)!}{(\ell+m)!}}\,(-)^m e^{im\varphi} P_{\ell m}(\cos\theta)$$

so that

$$Y_{\ell 0} = \sqrt{\frac{2\ell+1}{4\pi}}\; P_\ell(\cos\theta)$$

$$Y_{\ell 1} = -\sqrt{\frac{2\ell+1}{4\pi}\frac{1}{\ell(\ell+1)}}\; e^{i\varphi} P_{\ell 1}(\cos\theta) .$$

It will be convenient to choose a normalization such that $N = \sqrt{2\pi(2j+1)}$ and we finally have, remembering that for χ_j^1, $j = \ell+\frac{1}{2}$ and for χ_j^2, $j = \ell - \frac{1}{2}$,

$$\chi_j^1 = (j+\tfrac{1}{2})P_{j-\frac{1}{2}}(\cos\theta)\alpha - e^{i\varphi}P_{j-\frac{1}{2},1}(\cos\theta)\beta , \qquad (6\text{-}20a)$$

$$\chi_j^2 = (j+\tfrac{1}{2})P_{j+\frac{1}{2}}(\cos\theta)\alpha + e^{i\varphi}P_{j+\frac{1}{2},1}(\cos\theta)\beta \quad , \qquad (6\text{-}20b)$$

or we may write

$$\chi_j^1 = \begin{pmatrix} (j+\tfrac{1}{2})P_{j-\frac{1}{2}}(\cos\theta) \\ \\ -e^{i\varphi}P_{j-\frac{1}{2},1}(\cos\theta) \end{pmatrix}, \qquad \chi_j^2 = \begin{pmatrix} (j+\tfrac{1}{2})P_{j+\frac{1}{2}}(\cos\theta) \\ \\ e^{i\varphi}P_{j+\frac{1}{2},1}(\cos\theta) \end{pmatrix}.$$

The appearance of two Legendre functions of order $j\pm\frac{1}{2}$ is due to the fact that for a given total angular momentum j, we can have two orbital angular momenta corresponding to the spin being parallel or anti-parallel to the orbital angular momentum.

The partial wave solutions of eq.(6-4a) then take the form

$$\phi = \sum a_j\phi_j = \sum_{j-\frac{1}{2}=0}^{\infty} (kr)^{-1}a_j\left[G_j(r)\chi_j^1 + iF_j(r)\chi_j^2\right] \qquad (6\text{-}21)$$

where $G_j(r)$ and $F_j(r)$ are radial wave functions still to be determined. The appearence of i in the last term is merely a matter of convenience. With the help of the recurrence relations[30] for the Legendre functions, we find that

$$\chi_j^1 + \chi_j^2 = (j+\tfrac{1}{2})(P_{j+\frac{1}{2}} + P_{j-\frac{1}{2}})\sec\frac{\theta}{2}\left[\cos\frac{\theta}{2}\alpha + \sin\frac{\theta}{2}e^{i\varphi}\beta\right], \qquad (6\text{-}22)$$

$$\chi_j^1 - \chi_j^2 = (j+\tfrac{1}{2})(P_{j+\frac{1}{2}} - P_{j-\frac{1}{2}})\operatorname{cosec}\frac{\theta}{2}e^{i\varphi}\left[-\sin\frac{\theta}{2}e^{-i\varphi}\alpha + \cos\frac{\theta}{2}\beta\right]. \qquad (6\text{-}23)$$

See also appendix J. If these are compared with equations (6-8), we see that eqs.(6-22) and (6-23) correspond to spin parallel and anti-parallel to the direction(θ,φ), respectively. In deriving the equations satistied by F_j and G_j, the following identities are useful:

$$\vec{\sigma}\cdot\vec{r}\chi_j^1 = r\chi_j^2 \quad , \tag{6-24a}$$

$$\vec{\sigma}\cdot\vec{r}\chi_j^2 = r\chi_j^1 \quad , \tag{6-24b}$$

$$\vec{\sigma}\cdot\vec{L}\chi_j^1 = \hbar(j-\tfrac{1}{2})\chi_j^1 \quad , \tag{6-24c}$$

$$\vec{\sigma}\cdot\vec{L}\chi_j^2 = -\hbar(j+3/2)\chi_j^2 \quad , \tag{6-24d}$$

$$r^2(\vec{\sigma}\cdot\vec{p}) = (\vec{\sigma}\cdot\vec{r})(\vec{r}\cdot\vec{p}) + i(\vec{\sigma}\cdot\vec{r})(\vec{\sigma}\cdot\vec{L}) \quad . \tag{6-24e}$$

(See the proof in Appendix J.) These expressions are now substituted into the Dirac equation (6-4a) to obtain radial wave equations satisfied by F_j and G_j. The Dirac equation is

$$(\vec{\sigma}\cdot\vec{p} + \frac{V-E}{c})\,\phi_{j,\frac{1}{2}} = 0$$

where $\phi_{j,\frac{1}{2}} = (kr)^{-1}a_j\{G_j(r)\chi_j^1 + iF_j(r)\chi_j^2\}$.

Hence we have

$$(\vec{\sigma}\cdot\vec{p})\phi_{j,\frac{1}{2}} = \frac{1}{r^2}\{(\vec{\sigma}\cdot\vec{r})(\vec{r}\cdot\vec{p}) + i(\vec{\sigma}\cdot\vec{r})(\vec{\sigma}\cdot\vec{L})\}$$

$$\times\frac{a_j}{kr}\{G_j(r)\chi_j^1 + iF_j(r)\chi_j^2\}$$

$$= \frac{a_j}{kr}\left[-i\hbar\left(\frac{dG_j}{dr} - \frac{j+\frac{1}{2}}{r}G_j\right)\chi_j^2 + \hbar\left(\frac{dF_j}{dr} + \frac{j+\frac{1}{2}}{r}F_j\right)\chi_j^1\right] .$$

This is equal to $\frac{E-V}{c}\phi_{j,\frac{1}{2}}$.

Thus we obtain

$$-i\hbar\left(\frac{dG_j}{dr} - \frac{j+\frac{1}{2}}{r}G_j\right)\chi_j^2 + \hbar\left(\frac{dF_j}{dr} + \frac{j+\frac{1}{2}}{r}F_j\right)\chi_j^1$$

$$= \frac{E-V}{c}\left(G_j\chi_j^1 + iF_j\chi_j^2\right) .$$

Now if V is spherically symmetric, we can separate the radial and angular parts and obtain the radial equations

$$\frac{dG_j}{dr} - \frac{j+\frac{1}{2}}{r}G_j + \frac{E-V}{\hbar c}F_j = 0 , \tag{6-25}$$

$$\frac{dF_j}{dr} + \frac{j+\frac{1}{2}}{r}F_j - \frac{E-V}{\hbar c}G_j = 0 . \tag{6-26}$$

For use in the scattering calculation, we need the solutions of eqs.(6-25) and (6-26) which are regular at the origin. It should be mentioned that we had the coupled differential equations (1-13a) and (1-13b) in chapter 1 and that if the electron mass is neglected, these equations reduce respectively to eqs.(6-25) and (6-26) except that the role of F and G is reversed.

Before going into the calculation of the scattering cross section, it may be useful to examine the form of the radial wave function for the potential $V = 0$, i.e., the free particle problem. The coupled first order differential equations (6-25) and (6-26) are readily separated into two second order differential equations, for example,

$$\frac{d^2G}{dr^2} + \left[- \frac{(j+\frac{1}{2})(j-\frac{1}{2})}{r^2} + \left(\frac{E}{\hbar c}\right)^2 \right] G = 0,$$

and this is further reduced to

$$\frac{d^2G}{d\rho^2} + \left[1 - \frac{(j+\frac{1}{2})(j-\frac{1}{2})}{\rho^2} \right] G = 0 , \tag{6-27}$$

where $\rho = kr$

As is well known[31], the solution of eq.(6-27) which is regular at the origin is the spherical Bessel function g_ℓ. The spherical Bessel function is related to an ordinary Bessel function of half-integer order $J_{\ell+\frac{1}{2}}$ by the expression

$$g_\ell(kr) = \sqrt{\frac{\pi}{2kr}}\; J_{\ell+\frac{1}{2}}(kr) . \tag{6-28}$$

For the convenience, we here give the asymptotic form of the spherical Bessel function

$$g_\ell(\rho) \underset{\rho\to\infty}{\sim} \frac{1}{\rho}\cos\left(\rho - \tfrac{1}{2}(\ell+1)\pi \right) . \tag{6-29}$$

From eq.(6-29), together with eq.(6-28), it is found that

$$\sqrt{\frac{\rho\pi}{2}}\; J_{\ell+\frac{1}{2}}(\rho) \underset{\rho\to\infty}{\sim} \cos\left(\rho - \tfrac{1}{2}(\ell+1)\pi \right). \tag{6-30}$$

Hence we find asymptotic forms of J_j for $j=\ell+\frac{1}{2}$

$$\sqrt{\frac{\pi kr}{2}}\, J_j(kr) \sim \sin(kr - \tfrac{1}{2}(j-\tfrac{1}{2})\pi) \quad , \qquad (6\text{-}31a)$$

and for $j=\ell-\frac{1}{2}$

$$\sqrt{\frac{\pi kr}{2}}\, J_{j+1}(kr) \sim -\cos(kr-\tfrac{1}{2}(j-\tfrac{1}{2})\pi) \quad . \qquad (6\text{-}31b)$$

When the potential V is present, the asymptotic behaviour of G_j and F_j must be

$$G_j(kr) \sim \sin\{kr - \tfrac{1}{2}(j-\tfrac{1}{2})\pi + \delta_j\} \quad , \qquad (6\text{-}32a)$$

$$F_j(kr) \sim -\cos\{kr - \tfrac{1}{2}(j-\tfrac{1}{2})\pi + \delta_j\} \quad , \qquad (6\text{-}32b)$$

where δ_j is the phase shift due to the potential. For the Coulomb scattering we have the additional term $y\cdot \ln 2pr$ so that

$$G_j(kr) \sim \sin\{kr + y\cdot\ln 2pr - \tfrac{1}{2}(j-\tfrac{1}{2})\pi + \delta_j\} \qquad (5\text{-}32c)$$

$$= \sin\{kr + y\cdot\ln 2pr - \tfrac{1}{2}(k-1)\pi + \delta_j\}$$

$$F_j(kr) \sim -\cos\{kr + y\cdot\ln 2pr - \tfrac{1}{2}(j-\tfrac{1}{2})\pi + \delta_j\} \qquad (5\text{-}32d)$$

$$= -\cos\{kr + y\cdot\ln 2pr - \tfrac{1}{2}(k-1)\pi + \delta_j\}$$

At large distances, as mentioned earlier, we expect to find an asymptotic solution of the form

$$\phi \sim \begin{pmatrix}1\\0\end{pmatrix} e^{ikz} + \begin{pmatrix}\cos\frac{\theta}{2}\\ \sin\frac{\theta}{2}e^{i\varphi}\end{pmatrix} f(\theta,\varphi)\,\frac{e^{ikr}}{r} \qquad (6\text{-}9a)$$

$$= \phi_{inc} + \phi_{scatt}. \qquad (6\text{-}33)$$

Making use of the known expansion of the plane wave

$$e^{ikz} = \sum_{\ell=0}^{\infty} (2\ell+1) i^{\ell} j_{\ell}(r) P_{\ell}(\cos\theta) \quad ,$$

where j_ℓ are spherical Bessel functions, it is easily seen, using eq.(6-28), that

$$\phi_{inc} = \sum_{j-\frac{1}{2}=0}^{\infty} \sqrt{\frac{\pi}{2kr}}\, i^{j-1} \left[J_j(r)\chi_j^1 + iJ_{j+1}\chi_j^2 \right] . \quad (6\text{-}34)$$

We note that the first term in the square brackets of eq.(6-34) corresponds to $\ell = j-\frac{1}{2}$ and the second term to $\ell = j+\frac{1}{2}$ and that the expansion is expressed in terms of the mixed parity partial waves.

Now let us change the eq.(6-34) in a more convenient form

$$\phi_{inc} = \sum_{j-\frac{1}{2}=0}^{\infty} \frac{1}{kr} i^{j-\frac{1}{2}} \left[\sqrt{\frac{\pi kr}{2}}\, J_j(kr)\chi_j^1 + i\sqrt{\frac{\pi kr}{2}}\, J_{j+1}(kr)\chi_j^2 \right] \quad (6\text{-}35)$$

so that we can make use of the asymptotic forms of the Bessel functions (6-31a) and (6-31b). We find that

$$\phi_{inc} \sim \sum_{j-\frac{1}{2}=0}^{\infty} \frac{1}{kr} i^{j-\frac{1}{2}} \left[\sin\{kr-\frac{\pi}{2}(j-\tfrac{1}{2})\}\chi_j^1 - i\cos\{kr-\frac{\pi}{2}(j-\tfrac{1}{2})\}\chi_j^2 \right] . \quad (6\text{-}36)$$

The asymptotic forms of G_j and F_j are now substituted into eq. (6-21) to yield

$$\phi \sim \sum_{j-\frac{1}{2}=0} \frac{a_j}{kr}\left[\sin\left\{kr-\frac{\pi}{2}(j-\tfrac{1}{2})+\delta_j\right\}\chi_j^1 - i\cos\left\{kr-\frac{\pi}{2}(j-\tfrac{1}{2})+\delta_j\right\}\chi_j^2\right] . \tag{6-37}$$

Equating eq.(6-37) to (6-33) and substituting ϕ_{inc}, eq.(6-36), we find that

$$\sum_{j-\frac{1}{2}=0}^{\infty} \frac{i^{j-\frac{1}{2}}}{kr}\left\{\sin\Theta\chi_j^1 - i\cos\Theta\chi_j^2\right\} + \begin{pmatrix}\cos\frac{\theta}{2}\\ \\ \sin\frac{\theta}{2}e^{i\varphi}\end{pmatrix} f(\theta,\varphi)\frac{e^{ikr}}{r}$$

$$= \sum_{j-\frac{1}{2}=0}^{\infty} \frac{a_j}{kr}\left[\sin(\Theta+\delta_j)\chi_j^1 - i\cos(\Theta+\delta_j)\chi_j^2\right] \tag{6-38}$$

where $\Theta = kr - \frac{\pi}{2}(j-\frac{1}{2})$.

The coefficients a_j may be determined from the requirement that there must be no incoming spherical wave, i.e., no term in exp(-ikr)/r. Hence we have

$$a_j = i^{j-\frac{1}{2}}\, e^{i\delta_j} \tag{6-39}$$

and then we can rewrite eq.(6-38) as

$$\sum_{j-\frac{1}{2}=0}^{\infty} \frac{i^{j-\frac{1}{2}}}{kr}\left[\frac{1}{2i}(e^{i\Theta}-e^{-i\Theta})\chi_j^1 - \frac{i}{2}(e^{i\Theta}+e^{-i\Theta})\chi_j^2\right] + \begin{pmatrix}\cos\frac{\theta}{2}\\ \\ \sin\frac{\theta}{2}e^{i\varphi}\end{pmatrix} f(\theta,\varphi)\frac{e^{ikr}}{r}$$

$$= \sum_{j-\frac{1}{2}=0}^{\infty} \frac{i^{j-\frac{1}{2}}}{kr}\left[\frac{1}{2i}(e^{i\Theta}-e^{-i\Theta})\chi_j^1 - \frac{i}{2}(e^{i\Theta}+e^{-i\Theta})\chi_j^2\right]$$

$$+ \sum \frac{i^{j-\frac{1}{2}}}{kr} \frac{1}{2i} e^{i\theta} (e^{2i\delta_j} - 1)(\chi_j^1 + \chi_j^2) \quad . \tag{6-40}$$

We thus find from eq.(6-40) that the scattered wave can be written

$$f(\theta,\varphi) \begin{pmatrix} \cos\frac{\theta}{2} \\ \\ \sin\frac{\theta}{2} e^{i\varphi} \end{pmatrix} \frac{e^{ikr}}{r} = \sum \frac{i^{j-\frac{1}{2}}}{kr} \frac{1}{2i} e^{i\theta} (e^{2i\delta_j} - 1)(\chi_j^1 + \chi_j^2) \quad . \tag{6-41}$$

Using the eqs.(6-22) and (6-41), we finally obtain

$$f(\theta,\varphi) = \frac{1}{2ik} \sum_{j-\frac{1}{2}=0}^{\infty} (e^{2i\delta_j} - 1)(j+\tfrac{1}{2})(P_{j+\frac{1}{2}} + P_{j-\frac{1}{2}}) \sec\frac{\theta}{2} \quad . \tag{6-42}$$

The result obtained has to be modified slightly for the Coulomb-like potentials. Because of the long range nature of these potentials, the plase shifts δ_j have to be replaced by $y \cdot \ln 2pr + \delta_j$ where $y = \alpha ZW/p$. The addition of this infinite but constant phase shift, however, has no effect on the scattering and the scattering amplitude $f(\theta,\varphi)$ is still given by the equation (6-42).

In order to get the idea that how many partial waves are necessary for the scattering of a particle at certain incident energy, let us consider a classical case of the scattering. The trajectory of the particle may be characterized by the impact parameter b as shown in Fig. 6.1, and the angular

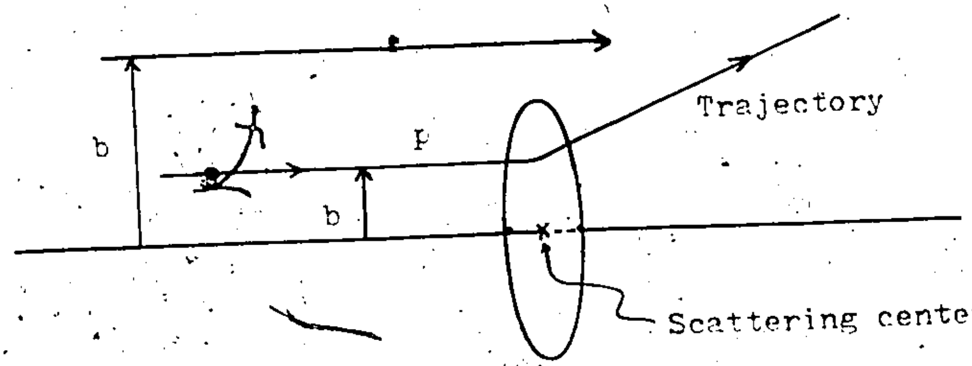

Fig.6.1 Impact parameter b for scattering of a classical particle.

momentum of the particle with this impact parameter is pb where p is the magnitude of the particle momentum. If the scattering has a finite range R, the scattering will take place only if $b<R$. The transition from the classical case to the quantum mechanical scattering can be made through the substitutions

$$pb \longrightarrow \hbar\{\ell(\ell+1)\}^{\frac{1}{2}}, \quad p \longrightarrow \hbar k$$

so that a finite contribution may be expected from those ℓ-th partial waves for which

$$\{\ell(\ell+1)\}^{\frac{1}{2}} < kR = \frac{ER}{\hbar c} \tag{6-43}$$

where $E = cp = c\hbar k$.

For a finite nucleus, the condition for the ℓ-values we must take into account can be determined by eq.(6-43) as

$$\{\ell(\ell+1)\}^{\frac{1}{2}} \sim kR .$$

To determine the effect of the finite nuclear size, it is customary that the scattering amplitudes are decomposed into the Coulomb amplitude and its deviation due to the finite nuclear size effect[26]. The scattering amplitude is usually calculated in the form

$$f(\theta) = f_c(\theta) + \{ f(\theta) - f_c(\theta) \} \tag{6-44}$$

where $f_c(\theta)$ is a scattering amplitude for a point nucleus with phase shifts δ_j^c and $f(\theta) - f_c(\theta)$ is given by the expression

$$f(\theta) - f_c(\theta) = \frac{1}{2ik} \sum (j+\tfrac{1}{2})(e^{2i\Delta_j} - e^{2i\delta_j^c})(P_{j-\frac{1}{2}} + P_{j+\frac{1}{2}}) \sec\frac{\theta}{2} . \tag{6-45}$$

Since the difference between Δ_j and δ_j^c tends to zero for large j ($j>kR$), the series (6-45) converges, even though Δ_j and δ_j^c do not separately tend to zero. However, since $f(\theta)$ is small at large angles the two terms of equation (6-44) are approximately equal but of opposite sign and hence considerable cancellations take place. It is therefore necessary to know $f_c(\theta)$ very accurately. Yennie[26] et.al. have been most successful in developing a method of quickly summing the series for $f(\theta)$. In older calculations the series converged very slowly, but they improved the evaluation of the sum considerably by introducing the method of their so-called "reduced" series. These series are formed by multiplying $f(\theta)$ by powers of $(1-\cos\theta)$. We represent $f(\Theta)$ by

$$2ikf(\theta) = \sum a_\ell P_\ell(\cos\theta) \quad . \tag{6-46}$$

The m-th "reduced" series is defined by

$$(1-\cos\theta)^m\, 2ikf(\theta) = \sum a_\ell^{(m)} P_\ell(\cos\theta) \quad . \tag{6-47}$$

From eq.(6-47), it is observed that

$$2ikf = \sum a_\ell^{(0)} P_\ell(\cos\theta) \qquad \text{for} \quad m=0 \,, \tag{6-48a}$$

and

$$(1-\cos\theta)2ikf = \sum a_\ell^{(1)} P_\ell(\cos\theta) \qquad \text{for} \quad m=1 \,. \tag{6-49b}$$

Using eq.(6-48a), we rewrite eq.(6-49b) as

$$\sum_{\ell=0} a_\ell^{(1)} P_\ell(\cos\theta) = \sum_{\ell=0} a_\ell^{(0)} P_\ell(\cos\theta) - \sum_{\ell=0} a_\ell^{(0)} \cos\theta\, P_\ell(\cos\theta)$$

$$= \sum_{\ell=0} a_\ell^{(0)} P_\ell(\cos\theta) - \sum_{\ell=0} a_\ell^{(0)} \frac{\ell+1}{2\ell+1} P_{\ell+1}(\cos\theta)$$

$$- \sum_{\ell=0} a_\ell^{(0)} \frac{\ell}{2\ell+1} P_{\ell-1}(\cos\theta) \tag{6-50}$$

where we have used the recurrence relation

$$(\ell+1)P_{\ell+1}(\cos\theta) - (2\ell+1)\cos\theta P_\ell(\cos\theta) + \ell P_{\ell-1}(\cos\theta) = 0 \quad .$$

Changing a dummy suffix ℓ of the second and the third term of eq.(6-50) into $\ell-1$ and $\ell+1$, respectively, we find

$$\sum_{\ell=0} a_\ell^{(1)} P_\ell(\cos\theta) = \sum_{\ell=0}\left\{a_\ell^{(0)} - \frac{\ell+1}{2\ell+3} a_{\ell+1}^{(0)} - \frac{\ell}{2\ell-1} a_{\ell-1}^{(0)}\right\} P_\ell(\cos\theta) .$$

We therefore have the recurrence relation for the coefficients $\bar{a_\ell}$.

$$a_\ell^{(1)} = a_\ell^{(0)} - \frac{\ell+1}{2\ell+3} a_{\ell+1}^{(0)} - \frac{\ell}{2\ell-1} a_{\ell-1}^{(0)}$$

For $m=2$, starting with

$$\sum a_\ell^{(2)} P_\ell(\cos\theta) = (1-\cos\theta)^2 \cdot 2ikf(\theta)$$

$$= (1-\cos\theta)\sum a_\ell^{(1)} P_\ell(\cos\theta) = \sum a_\ell^{(1)} - \sum a_\ell^{(1)} \cos\theta\, P_\ell(\cos\theta) ,$$

we readily find the recurrence relation for $a_\ell^{(2)}$. The same procedure will be applied to yield any higher order of $a_\ell^{(m)}$ and, in general,

$$a_\ell^{(i+1)} = a_\ell^{(i)} - \frac{\ell+1}{2\ell+3} a_{\ell+1}^{(i)} - \frac{\ell}{2\ell-1} a_{\ell-1}^{(i)} \quad . \qquad (6-51)$$

The reduced series converge rapidly and so $f(\theta)$ can be found quickly by dividing the reduced sum by the power of $(1-\cos\theta)$.

CHAPTER 7

RELATIONS BETWEEN PHASES USED IN BETA DECAY AND ELECTRON SCATTERING

The aim of this chapter is to understand that how the phase shifts used in the work of YRW[29] are related to that of BR[4,5]. The former has been extensively used in the analysis of the high energy electron scattering and the latter, on the other hand, in the analysis of β-decay. In this chapter we intend to apply the theory developed in part I to the scattering problem. The most crucial point lies in eq. (2-46) where all informations concerning the nucleus are contained in the phase shift and what we should do is to find a way to rewrite eq.(2-46) in the YRW representation.

As was mentioned by Rose[8], the phase shifts derived in chapter 2

$$\delta_\kappa = -\arg\Gamma(\gamma+iy) + \eta - \frac{\pi\gamma}{2} \tag{2-29}$$

should be replaced by

$$\delta_\kappa = -\arg\Gamma(\gamma+iy) + \eta - \frac{\pi\gamma}{2} + \frac{1}{2}(\ell+1)\pi . \tag{7-1}$$

for the scattering use. The reason for the additional term $(\ell+1)\pi/2$ may be seen in the following: the phase shifts for the vanishing potential, which we shall denote by $\delta_\kappa(Z=0)$, are given by

$$\delta_\kappa(Z=0) = -\frac{1}{2}(\ell+1)\pi \tag{7-2}$$

since, when Z=0 we have, using eq.(2-29) with eq.(2-20),

$$\delta_\kappa(Z=0) = \begin{cases} -\dfrac{\pi|\kappa|}{2} = -\dfrac{\ell+1}{2}\pi & \text{for } \kappa<0 \\[2ex] -\dfrac{\pi}{2}-\dfrac{\pi|\kappa|}{2} = -\dfrac{\ell+1}{2}\pi & \text{for } \kappa>0 \end{cases}$$

where η is taken to lie between $-\pi/2$ and $+\pi/2$. Thus the value of the "external" phase shifts δ_κ of eq.(7-1) is the difference between the Coulomb phase shifts exclusive of the logarithmic term and the phase shifts $\delta_\kappa(Z=0)$. In other words, there is the additional phase caused by the Coulomb field. To further confirm this situation, we shall begin with the YRW expression. The Coulomb phase shifts used in YRW are given by the equation

$$e^{2i\delta_j^c} = \frac{\gamma-iy}{j+\frac{1}{2}}\frac{\Gamma(\gamma-iy)}{\Gamma(\gamma+iy)}e^{\pi i(j+\frac{1}{2}-\gamma)}$$

$$= \frac{\gamma-iy}{k}\frac{\Gamma(\gamma-iy)}{\Gamma(\gamma+iy)}e^{\pi i(k-\gamma)} \tag{7-3}$$

where $k = j+\frac{1}{2}>0$ and $y = \alpha ZW/p$. We rewrite eq.(7-3) as

$$e^{2i\delta_j^c} = \frac{\gamma-iy}{k}\frac{|\Gamma(\gamma-iy)|\,e^{i\arg\Gamma(\gamma-iy)}}{|\Gamma(\gamma+iy)|\,e^{i\arg\Gamma(\gamma+iy)}}e^{\pi i(k-\gamma)}$$

$$= \frac{\gamma-iy}{k}e^{2i\left\{-\arg\Gamma(\gamma+iy)-\frac{\pi}{2}\gamma+\frac{\pi}{2}k\right\}} \tag{7-4}$$

As we have seen, η_κ satisfies the equation

$$e^{2i\eta_\kappa} = -\frac{\kappa - i\frac{y}{W}}{\gamma + iy} = -\frac{(\gamma - iy)(\kappa - i\frac{y}{W})}{\kappa^2 - (\alpha Z)^2 + y^2} . \qquad (2\text{-}20)$$

In the limit of the infinite energy $W(=p)$, the equation above tends to

$$e^{2i\eta_\kappa} \longrightarrow -\frac{\gamma - iy}{\kappa} \qquad (7\text{-}5)$$

since y is approaching (αZ) as is W increasing. Thus, depending on the sign of κ, we have

$$e^{2i\eta_\kappa} \longrightarrow \begin{cases} \dfrac{\gamma - iy}{k} & \text{for } \kappa < 0 . \qquad (7\text{-}6a) \\[2ex] -\dfrac{\gamma - iy}{k} & \text{for } \kappa > 0 . \qquad (7\text{-}6b) \end{cases}$$

For the irregular solutions, γ is replace by $-\gamma$ so that we have

$$e^{2i\bar{\eta}_\kappa} \longrightarrow \begin{cases} -\dfrac{\gamma + iy}{k} & \text{for } \kappa < 0 . \qquad (7\text{-}6c) \\[2ex] \dfrac{\gamma + iy}{k} & \text{for } \kappa > 0 . \qquad (7\text{-}6d) \end{cases}$$

We obtain, from eqs. (7-4) and (7-6a), for $\kappa < 0$,

$$\delta_j^c = -\arg\Gamma(\gamma + iy) - \frac{\pi\gamma}{2} + \eta_\kappa + \frac{\pi}{2}(j + \tfrac{1}{2})$$

$$= -\arg\Gamma(\gamma + iy) - \frac{\pi\gamma}{2} + \eta_\kappa + \frac{\pi}{2}k , \qquad (7\text{-}7a)$$

and using $(-1) = e^{2i(\frac{\pi}{2})}$, we obtain, for $\kappa > 0$,

$$\delta_j^c = -\arg\Gamma(\gamma+iy) - \frac{\pi\gamma}{2} + \eta_\kappa + \frac{\pi}{2}(j+3/2)$$

$$= -\arg\Gamma(\gamma+iy) - \frac{\pi\gamma}{2} + \eta_\kappa + \frac{\pi}{2}(k+1). \qquad (7\text{-}7b)$$

Both (7-7a) and (7-7b) are expressed in a single form

$$\delta_j^c = -\arg\Gamma(\gamma+iy) - \frac{\pi\gamma}{2} + \eta_\kappa + \frac{\pi}{2}(\ell+1). \qquad (7\text{-}7c)$$

Here we have used the fact that the angular momentum j is equal to $\ell+\frac{1}{2}$ and $\ell-\frac{1}{2}$ for $\kappa<0$ and for $\kappa>0$, respectively. Starting from the YRW expression in the high energy limit, we have the additional phase $\pi(\ell+1)/2$ in eq.(7-7c) compared with eq.(2-29).

The phase shift Δ_j which is given by eq.(37) of YRW is expressed in the form

$$\tan(\Delta_j - \delta_j^c) = \frac{\sin(\bar{\delta}_j^c - \delta_j^c)}{B/C + \cos(\bar{\delta}_j^c - \delta_j^c)}, \qquad (7\text{-}8)$$

while in BR's expression, eq.(2-48),

$$\tan\Delta_\kappa = \frac{\sin\delta_\kappa + \frac{C}{B}\sin\bar{\delta}_\kappa}{\cos\delta_\kappa + \frac{C}{B}\cos\bar{\delta}_\kappa}. \qquad (7\text{-}9)$$

Let us rewrite eq.(7-8) by using identity

$$\tan(\Delta_j - \delta_j^c) = \frac{\tan\Delta_j - \tan\delta_j^c}{1 + \tan\Delta_j\tan\delta_j^c}. \qquad (7\text{-}10)$$

The right hand side of eq.(7-8) is written as

$$\frac{\sin(\overline{\delta}_j^c - \delta_j^c)}{\frac{B}{C} + \cos(\overline{\delta}_j^c - \delta_j^c)} = \frac{\tan\overline{\delta}_j^c - \tan\delta_j^c}{1 + \frac{B}{C}\frac{1}{\cos\delta_j^c\cos\overline{\delta}_j^c} + \tan\delta_j^c\tan\overline{\delta}_j^c} \quad (7\text{-}11)$$

Equating eq.(7-10) with eq.(7-11), we obtain.

$$\tan\Delta_j = \frac{\sin\delta_j^c + \frac{C}{B}\sin\overline{\delta}_j^c}{\cos\delta_j^c + \frac{C}{B}\cos\overline{\delta}_j^c} \quad . \quad (7\text{-}12)$$

Thus comparing eq.(7-12) with eq.(7-9), with admitting the difference in the notation, we see that the phase shift used in BR coinsides with the YRW's. The change made for the scattering in BR is, therefore, the one that enters eq.(7-1) as the additional phase

$$\frac{1}{2}(\ell+1)\pi \quad . \quad (7\text{-}13)$$

We have seen that the asymptotic radial wave functions F^c and G^c are of the form

$$F^c \sim -\sin(pr + y\cdot\ln 2pr + \delta_\kappa^R) \quad (2\text{-}26)$$

and

$$G^c \sim \cos(pr + y\cdot\ln 2pr + \delta_\kappa^R) \quad (2\text{-}27)$$

here we put the superscript R to specify the phase shifts in the BR representation, and δ_κ^R is given by

$$\delta_\kappa^R = -\arg\Gamma(\gamma+iy) + \eta_\kappa - \frac{\pi\gamma}{2} \quad . \quad (2\text{-}29)$$

Similarly it is necessary to express the YRW asymptotic wave functions as a function of sin or cos where the phase shifts δ_κ^Y of YRW representation must be different from δ_κ^R, because their asymptotic functions have the additional term $-\frac{1}{2}(j-\frac{1}{2})\pi$ as was shown in eqs.(6-32c) and (6-32d).

Since the scattering amplitude $f(\theta)$ used in determining the cross section of the electron scattering was derived by using the YRW representation, it is necessary to modify the BR method, especially the phase shifts δ_κ^R, but this becomes complicated. The reason why this is complicated is that the asymptotic radial wave functions F^c and G^c change their roles due to the trigonometric function involved, which also changes in its form from one to another depending on the value of k, as will be seen shortly.

Hereafter we clearly distinguish the negative value of κ (or Z) from its positive value by adding the bar beneath it to denote the negative quantity, for example if $\kappa > 0$, we write κ as before and if $\kappa < 0$, we write $\underline{\kappa}$ instead. Thus in BR expression, we write

$$\delta_{\underline{\kappa}}^R = -\arg\Gamma(\gamma+iy) + \eta_{\underline{\kappa}} - \frac{\pi\gamma}{2}\,, \tag{7-14a}$$

$$\delta_{\kappa}^R = -\arg\Gamma(\gamma+iy) + \eta_{\kappa} - \frac{\pi\gamma}{2}\,, \tag{7-14b}$$

$$\bar{\delta}_{\underline{\kappa}}^R = -\arg\Gamma(-\gamma+iy) + \bar{\eta}_{\underline{\kappa}} + \frac{\pi\gamma}{2}\,, \tag{7-14c}$$

and

$$\bar{\delta}_{\kappa}^R = -\arg\Gamma(-\gamma+iy) + \bar{\eta}_{\kappa} + \frac{\pi\gamma}{2}\,, \tag{7-14d}$$

where η satisfies

$$\tan \eta_{\underline{\kappa}} = \frac{\alpha Z}{p} \frac{W-1}{\underline{\kappa}-\gamma} , \qquad (7\text{-}15a)$$

$$\tan \bar{\eta}_{\underline{\kappa}} = \frac{\alpha Z}{p} \frac{W-1}{\underline{\kappa}+\gamma} , \qquad (7\text{-}15b)$$

$$\tan \eta_{\kappa} = \frac{\alpha Z}{p} \frac{W-1}{\kappa+\gamma} , \qquad (7\text{-}15c)$$

and

$$\tan \bar{\eta}_{\kappa} = \frac{\alpha Z}{p} \frac{W-1}{\kappa-\gamma} . \qquad (7\text{-}15d)$$

(See also Appendix K for an alternative method to determine η_κ in the high energy limit.)

Whereas in YRW expression, from eqs.(7-7a) and (7-7b), we have

$$\delta^{Y}_{\underline{\kappa}} = -\arg\Gamma(\gamma+iy) - \frac{\pi\gamma}{2} + \eta_{\underline{\kappa}} + \frac{\pi}{2}k , \qquad (7\text{-}16a)$$

$$\delta^{Y}_{\kappa} = -\arg\Gamma(\gamma+iy) - \frac{\pi\gamma}{2} + \eta_{\kappa} + \frac{\pi}{2}(k+1) , \qquad (7\text{-}16b)$$

$$\bar{\delta}^{Y}_{\underline{\kappa}} = -\arg\Gamma(-\gamma+iy) + \frac{\pi\gamma}{2} + \bar{\eta}_{\underline{\kappa}} - \frac{\pi}{2}k , \qquad (7\text{-}16c)$$

and

$$\bar{\delta}^{Y}_{\kappa} = -\arg\Gamma(-\gamma+iy) + \frac{\pi\gamma}{2} + \bar{\eta}_{\kappa} - \frac{\pi}{2}(k+1) , \qquad (7\text{-}16d)$$

See Appendix L for the Coulomb phase shift for the irregular solution. Thus we may write $\delta^{R}_{\underline{\kappa}}$ as

$$\delta^{R}_{\underline{\kappa}} = -\arg\Gamma(\gamma+iy) - \frac{\pi\gamma}{2} - \frac{\pi}{2}(k-1) + \eta_{\underline{\kappa}} + \frac{\pi}{2}(k-1).$$

If we denote the whole argument of the asymptotic wave functions by Ω, then we have

$$\Omega^R_{\underline{\kappa}} = pr + y\cdot\ln 2pr + \delta^R_{\underline{\kappa}}$$

$$= pr + y\cdot\ln 2pr - \arg\Gamma(\gamma+iy) - \frac{\pi\gamma}{2} - \frac{\pi}{2}(k-1) + \eta_{\underline{\kappa}} + \frac{\pi}{2}(k-1)$$

$$= \omega + \left(\eta_{\underline{\kappa}} + \frac{\pi}{2}(k-1)\right) , \qquad (7\text{-}17a)$$

and

$$\Omega^Y_{\underline{\kappa}} = pr + y\cdot\ln 2pr - \frac{\pi}{2}(k-1) + \delta^Y_{\underline{\kappa}}$$

$$= pr + y\cdot\ln 2pr - \frac{\pi}{2}(k-1) - \arg\Gamma(\gamma+iy) - \frac{\pi\gamma}{2} + \eta_{\underline{\kappa}} + \frac{\pi}{2}k$$

$$= \omega + \left(\eta_{\underline{\kappa}} + \frac{\pi}{2}k\right) \qquad (7\text{-}17b)$$

where

$$\omega = pr + y\cdot\ln 2pr - \arg\Gamma(\gamma+iy) - \frac{\pi\gamma}{2} - \frac{\pi}{2}(k-1)$$

which is the common term in Ω^R and Ω^Y. Similarly we have

$$\bar{\Omega}^R_{\underline{\kappa}} = \omega + \bar{\eta}_{\underline{\kappa}} + \frac{\pi}{2}(k-1) , \qquad (7\text{-}17c)$$

$$\bar{\Omega}^Y_{\underline{\kappa}} = \omega + \bar{\eta}_{\underline{\kappa}} - \frac{\pi}{2}k , \qquad (7\text{-}17d)$$

$$\Omega^R_{\kappa} = \omega + \eta_{\kappa} + \frac{\pi}{2}(k-1) , \qquad (7\text{-}17e)$$

$$\Omega^Y_{\kappa} = \omega + \eta_{\kappa} + \frac{\pi}{2}(k+1) , \qquad (7\text{-}17f)$$

$$\bar{\Omega}^R_{\kappa} = \omega + \bar{\eta}_{\kappa} + \frac{\pi}{2}(k-1) , \qquad (7\text{-}17g)$$

and $$\bar{\Omega}^Y_{\kappa} = \omega + \bar{\eta}_{\kappa} - \frac{\pi}{2}(k+1) . \qquad (7\text{-}17h)$$

Table 7.1 Variation of the tangent for $\kappa > 0$

$\bar{\eta}_\kappa \longrightarrow \bar{\eta}_\kappa + \frac{\pi}{2}(k-1)$		k			
		1	2	3	4
Rose:	$\tan(\bar{\eta}_\kappa + \frac{\pi}{2}(k-1))$	$\tan\bar{\eta}_\kappa$	$-\cot\bar{\eta}_\kappa$	$\tan\bar{\eta}_\kappa$	$-\cot\bar{\eta}_\kappa$
Yennie:	$\tan(\bar{\eta}_\kappa - \frac{\pi}{2}(k+1))$	$\tan\bar{\eta}_\kappa$	$-\cot\bar{\eta}_\kappa$	$\tan\bar{\eta}_\kappa$	$-\cot\bar{\eta}_\kappa$

Table 7.2 Variation of the tangent for $\kappa < 0$

$\eta_{\underline{\kappa}} \longrightarrow \eta_{\underline{\kappa}} + \frac{\pi}{2}(k-1)$		k			
		1	2	3	4
Rose:	$\tan(\eta_{\underline{\kappa}} + \frac{\pi}{2}(k-1))$	$\tan\eta_{\underline{\kappa}}$	$-\cot\eta_{\underline{\kappa}}$	$\tan\eta_{\underline{\kappa}}$	$-\cot\eta_{\underline{\kappa}}$
Yennie:	$\tan(\eta_{\underline{\kappa}} + \frac{\pi}{2}k)$	$-\cot\eta_{\underline{\kappa}}$	$\tan\eta_{\underline{\kappa}}$	$-\cot\eta_{\underline{\kappa}}$	$\tan\eta_{\underline{\kappa}}$

Thus from eqs.(7-17a) and (7-17b), we see that we may have to replace the original $\eta_{\underline{\kappa}}$ of BR by $(\eta_{\underline{\kappa}} + \frac{\pi}{2}(k-1))$ for the scattering but this is not good enough since $\Omega^{Y}_{\underline{\kappa}}$ itself changes according to the value of k. We need to know the relevant relations between the two. For instance, if $\bar{\eta}_\kappa$ is replaced by $\bar{\eta}_\kappa + \frac{\pi}{2}(k-1)$, then $\tan\bar{\eta}_\kappa$ of eq.(7-15d) must be replaced by $\tan(\bar{\eta}_{\kappa}+\frac{\pi}{2}(k-1))$ and at the same time we have to examine the property of $\tan(\bar{\eta}_{\kappa}-\frac{\pi}{2}(k+1))$ which comes into eq.(7-17h). Table 7.1 and 7.2 are prepared to see how these functions change depending on the value k up to 4. As is clearly seen, for $\kappa>0$, we can use eq.(7-15c) and (7-15d) to determine η_{κ} because both have the same trend. The situation, however, is different for $\kappa<0$. For the scattering case, for odd number of k, we have to use $\eta_{\underline{\kappa}}$ which satisfies the inverse of eq.(7-15a) times minus 1, and for even number of k, equation (7-15a) is resumed. Now we calculate $\cos\bar{\eta}/\cos\eta$. From eq. (2-21d), we obtain

$$\left(\frac{\cos\bar{\eta}}{\cos\eta}\right)_{\underline{\kappa}}=\left[\frac{(\underline{\kappa}+\gamma)(\underline{\kappa}W-\gamma)}{(\underline{\kappa}-\gamma)(\underline{\kappa}W+\gamma)}\right]^{\frac{1}{2}}, \qquad (7\text{-}18a)$$

and

$$\left(\frac{\cos\bar{\eta}}{\cos\eta}\right)_{\kappa}=\left[\frac{(\kappa-\gamma)(\kappa W+\gamma)}{(\underline{\kappa}+\gamma)(\underline{\kappa}W-\gamma)}\right]^{\frac{1}{2}}=1\Big/\left(\frac{\cos\bar{\eta}}{\cos\eta}\right)_{\underline{\kappa}} \qquad (7\text{-}18b)$$

Thus once the ratio of $\cos\bar{\eta}$ to $\cos\eta$ is known for $\kappa<0$, its inverse gives the ratio for $\kappa>0$. Using eqs.(7-15a), (7-15b), (7-18a) and (7-18b), we obtain

$$\left(\frac{\sin\bar{\eta}}{\sin\eta}\right)_{\underline{\kappa}}=\left[\frac{(\underline{\kappa}-\gamma)(\underline{\kappa}W-\gamma)}{(\underline{\kappa}+\gamma)(\underline{\kappa}W+\gamma)}\right]^{\frac{1}{2}}, \qquad (7\text{-}19a)$$

and similarly we find

$$\left(\frac{\sin\bar{\eta}}{\sin\eta}\right)_{\kappa}=1\Big/\left(\frac{\sin\bar{\eta}}{\sin\eta}\right)_{\underline{\kappa}} \qquad (7\text{-}19b)$$

We shall now modify $H_{\underline{\kappa}}$ of eq.(2-44) in YRW representation. Since $H_{\underline{\kappa}}$ consist of the ratios of one radial wave functions to another, we rather make modifications on the continuity equations of the radial wave functions at $r = \rho$ (see eqs.(2-42a) and (2-42b)), instead of inquiring how the individual radial wave function changes with respect to k. It is found that for odd number of k, the continuity equations are written

$$BG^c_{\underline{\kappa}} - C\overline{G}^c_{\underline{\kappa}} = AG^{(i)}_{\underline{\kappa}} , \tag{7-20a}$$

and

$$BF^c_{\underline{\kappa}} - C\overline{F}^c_{\underline{\kappa}} = AF^{(i)}_{\underline{\kappa}} . \tag{7-20b}$$

From these equations, eliminating A, we obtain

$$C = \frac{F^c_{\underline{\kappa}}/G^c_{\underline{\kappa}} - F^{(i)}_{\underline{\kappa}}/G^{(i)}_{\underline{\kappa}}}{\overline{F}^c_{\underline{\kappa}}/\overline{G}^c_{\underline{\kappa}} - F^{(i)}_{\underline{\kappa}}/G^{(i)}_{\underline{\kappa}}} \, \frac{G^c_{\underline{\kappa}}}{\overline{G}^c_{\underline{\kappa}}} \, B . \tag{7-20c}$$

We thus find that

$$H_{\underline{\kappa}} = \frac{F^c_{\underline{\kappa}}/G^c_{\underline{\kappa}} - F^{(i)}_{\underline{\kappa}}/G^{(i)}_{\underline{\kappa}}}{\overline{F}^c_{\underline{\kappa}}/\overline{G}^c_{\underline{\kappa}} - F^{(i)}_{\underline{\kappa}}/G^{(i)}_{\underline{\kappa}}} \, \frac{G^c_{\underline{\kappa}}}{\overline{G}^c_{\underline{\kappa}}} \quad \text{for odd k.} \tag{7-21}$$

Similarly we find that for even k, the continuity equation is

$$BF^c_{\underline{\kappa}} + C\overline{F}^c_{\underline{\kappa}} = AF^{(i)}_{\underline{\kappa}} \tag{7-22a}$$

and

$$BG^c_{\underline{\kappa}} + C\overline{G}^c_{\underline{\kappa}} = AG^{(i)}_{\underline{\kappa}} \tag{7-22b}$$

and that $H_{\underline{k}}$ has exactly the same form as was described in eq.(2-44)

$$H_{\underline{k}} = \frac{F^c_{\underline{k}}/G^c_{\underline{k}} - F^{(i)}_{\underline{k}}/G^{(i)}_{\underline{k}}}{F^{(i)}_{\underline{k}}/G^{(i)}_{\underline{k}} - \overline{F}^c_{\underline{k}}/\overline{G}^c_{\underline{k}}} \; \frac{G^c_{\underline{k}}}{\overline{G}^c_{\underline{k}}} \quad \text{for even k.} \qquad (2\text{-}23)$$

CHAPTER 8

DISCUSSION AND CONCLUSION

Using the method developed in the previous chapter, we have calculated the phase shifts of the regular Coulomb functions δ_j^c and the additional phase shifts $(\Delta_j - \delta_j^c)$ caused by the uniformly charged nucleus. Table 8.1 shows the values of phase shifts for gold, where the mass of the electron is neglected. The values of the YRW in Table 8.1 are taken from the reference 29) in which only positive κ's are taken into account. Small differences between YRW and ours may be caused by the slightly different values of parameters used. Table 8.2 represents the phase shifts for the same nucleus, where the electron mass is explicitly included in the calculation, indicating that there is in fact a difference in phase shifts between the case $m_e=0$ and $m_e \neq 0$ even for a higher Z value. The neglect of the electron mass gives rise to the significant inaccuracy[32)] in the calculation of the scattering cross section for a comparatively low energy ($E \lesssim 60$ MeV) electron by light nuclei ($Z \lesssim 22$).

Since our method of calculation gives reliable values of phase shifts, we are now in position to calculate the scattering cross section. To calculate the scattering cross section we postulate a charge distribution for the nuclear charge and then perform a very lengthy calculation on the

Table 8.1 Values of phase shifts for gold.*

	YRW ($\kappa>0$)		CURS ($\kappa>0$)		($\kappa<0$)	
	δ^c	$\Delta-\delta^c$	δ^c	$\Delta-\delta^c$	δ^c	$\Delta-\delta^c$
1	0.40736	-0.85820	0.40604	-0.85714	0.40865	-0.85958
2	-0.23797	-0.27143	-0.23862	-0.27159	-0.23713	-0.27258
3	-0.53303	-0.07633	-0.53347	-0.07598	-0.53260	-0.07636
4	-0.72659	-0.01494	-0.72691	-0.01491	-0.72626	-0.01501
5	-0.87098	-0.00199	-0.87123	-0.00196	-0.87071	-0.00198
6	-0.98623	-0.00017	-0.98644	-0.00019	-0.98600	-0.00019
7	-1.08218	-0.00001	-1.10824	-0.00001	-1.08198	-0.00001
8	-1.16438	-0.00000	-1.16452	-0.00000	-1.16420	-0.00000
9	-1.23628	-0.00000	-1.23641	-0.00000	-1.23612	-0.00000

* The electron mass is neglected here, and the incident energy of the electron is taken to be 113.12 MeV in our calculation.

Table 8.2 Values of $(\Delta - \delta_j^c)$ for gold.*

	$\kappa>0$	$\kappa<0$
1	-.85680	-.85925
2	-.27109	-.27209
3	-.07594	-.07631
4	-.01484	-.01493
5	-.00196	-.00197
6	-.00019	-.00019
7	-.00001	-.00001
8	-.00000	-.00000

* The electron mass is explicitly included in this calculation.

computer. In the historical point of view of the electron scattering analysis, two parameter shape of the Fermi and of the Gaussian charge distributions inside of a nucleus first seemed to fit the experimental data and, as the accuracy of the experiment was improved, these two distributions were also refined into three parameter shape as well as the FORTRAN code was entirely written in double precision. As far as these distributions are concerned, a possible conclusion might be drawn; [33,34] that is, the fit is sensitive to the amount of charge at large nuclear radii and there is a tendency for shapes in which there is less charge in the tail to fit the experiments better.

Should the diffused charge distribution studied in this work be closer to reality? Because of the fact that the diffused charge distribution does not have a long tail, compared with the Fermi distribution, as was shown in Fig 3.1, it is expected that our proposed charge distribution could facilitate the analysis of the electron scattering. We have calculated the elastic scattering cross section with the incident electron energy 153 MeV for ^{197}Au. In the computation we varied two parameters, i.e., the radial constant r_0 and the thickness t to minimize the square deviation function

$$D = \frac{1}{2}\sum\left[\sigma_{cal}(r_0,t;\theta,E) - \sigma_{exp}(\theta,E)\right]^2 \Big/ \left[\Delta\sigma(\theta,E)\right]^2 ,$$

and we included the partial waves and the reduced series up

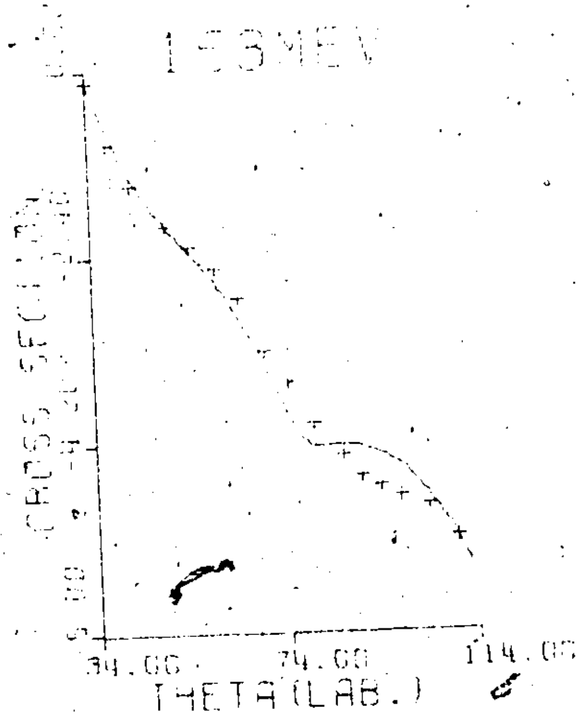

Fig 8.1 Cross sections at 153 MeV for scattering by the diffused shape $t=0.10$ fermi, for gold. Experimental values are indicated by +.

to $\ell=15$ and $m=3$, respectively. Fig 8.1 shows the calculated cross section vs experiment. The best fit occurs around $r_0=1.18$ and $t=0.10$ fermi and the square deviation being determined as 34.8. The nuclear radius R and the r.m.s. radius for this particular set of parameters are 6.87 fermi and 5.32 fermi, respectively. The value of "t" derived from this analysis is supprisingly small on the contrary to the Fermi distribution whose thickness* is about 2.3 fermi, although the nuclear radius R and the radial constant r_0 obtained are in good agreement with those derived through the Fermi distribution.

We have calculated the cross sections for another element ^{88}Sr where the incident electron energy 200 MeV being used. The results are shown in Table 8.3 together with the experimental data[34]. Although the results obtained do not give the complete satisfaction, it is hoped that the diffused charge distribution can be improved by making the sharp edge rounded and/or by using a slightly inclined charge distribution in the central region.

---

* Definitions of the "thickness" of the diffused and the Fermi distribution are not the same. A direct comparison may be difficult.

Table 8.3 Cross sections at 200 MeV.

The radial constant r_0 is assumed to be 1.18 and the thickness t is varied.

	Angle(deg)	40.9	48.8	55.9	65.5	77.4	97.1	111.7
σ_{exp}		.2203	$.2484\ 10^{-1}$	$.1321\ 10^{-1}$	$.5767\ 10^{-2}$	$.8361\ 10^{-3}$	$.1800\ 10^{-4}$	$.1353\ 10^{-4}$
	t=0.76	.2203	$.4043\ 10^{-1}$	$.1035\ 10^{-1}$	$.2366\ 10^{-2}$	$.6531\ 10^{-3}$	$.8509\ 10^{-4}$	$.5891\ 10^{-5}$
σ_{cal}	t=0.80	.2203	$.4290\ 10^{-1}$	$.1165\ 10^{-1}$	$.2772\ 10^{-2}$	$.6935\ 10^{-3}$	$.8226\ 10^{-4}$	$.8966\ 10^{-5}$
	t=0.88	.2203	$.4527\ 10^{-1}$	$.1291\ 10^{-1}$	$.3147\ 10^{-2}$	$.7236\ 10^{-3}$	$.8205\ 10^{-4}$	$.1353\ 10^{-4}$

APPENDICES

APPENDIX A

CHARGE DISTRIBUTION AND POTENTIAL

Potential associated with the spherically symmitric charge distribution is written as

$$V(r) = -4\pi Ze^2 \left[\frac{1}{r} \int_0^r \rho_e(r')r'^2 dr' + \int_r^\infty \rho_e(r')r' dr' \right] . \tag{1-3}$$

This can be proved in the following way: for a spherically symmetric charge distribution $\rho_e(r)$, the Coulomb potential is given by

$$V(r) = -Ze^2 \int \frac{\rho_e(r')}{|\vec{r} - \vec{r}'|} d\vec{r}' .$$

Now

$$\frac{1}{|\vec{r} - \vec{r}'|} = \frac{1}{\sqrt{r^2 - 2rr'\cos\theta + r'^2}}$$

$$= \begin{cases} \frac{1}{r}\left\{1 - 2\frac{r'}{r}\cos\theta + \left(\frac{r'}{r}\right)^2\right\}^{-\frac{1}{2}} \\ \quad = \frac{1}{r}\sum_{l=0}\left(\frac{r'}{r}\right)^\ell P_l(\cos\theta) \quad \text{for} \quad r'<r \\ \frac{1}{r'}\left\{1 - 2\frac{r}{r'}\cos\theta + \left(\frac{r}{r'}\right)^2\right\}^{-\frac{1}{2}} \\ \quad = \frac{1}{r'}\sum_{l=0}\left(\frac{r}{r'}\right)^\ell P_l(\cos\theta) \quad \text{for} \quad r'>r \end{cases}$$

where we have used the fact

$$\frac{1}{\sqrt{1 - 2sx + s^2}} = \sum_{\ell=0}^{\infty} s^{\ell} P_{\ell}(x) \qquad \text{for} \quad s < 1 .$$

Therefore

$$\int \frac{\rho_e(r')d\vec{r}'}{|\vec{r} - \vec{r}'|}$$

$$= \iint_{\Omega} \int_0^r \frac{1}{r} \sum_{\ell=0}^{\infty} \left(\frac{r'}{r}\right)^{\ell} P_{\ell}(\cos\theta)\, \rho e(r') r'^2 dr' d\Omega$$

$$+ \iint_{\Omega} \int_r \frac{1}{r'} \sum_{\ell=0}^{\infty} \left(\frac{r}{r'}\right)^{\ell} P_{\ell}(\cos\theta)\, \rho e(r') r'^2 dr' d\Omega \qquad \text{(A-1)}$$

Integration of the first term of (A-1) can be performed as follows: using

$$P_{\ell}(\cos\theta) = \left(\frac{4\pi}{2\ell + 1}\right)^{\frac{1}{2}} Y_{\ell 0}(\theta,\varphi), \text{ and } Y_{00} = \frac{1}{\sqrt{4\pi}} ,$$

we then have

$$\int Y_{\ell m} d\Omega = \int 1 \cdot Y_{\ell m} d\Omega$$

$$= \sqrt{4\pi} \int Y_{00}^{*} Y_{\ell m} d\Omega = \sqrt{4\pi}\, \delta_{\ell 0} \delta_{m0}$$

Therefore

$$\iint_{\Omega} \int_0^r \frac{1}{r} \sum_{\ell=0}^{\infty} \left(\frac{r'}{r}\right)^{\ell} P_{\ell}(\cos\theta)\, \rho_e(r') r'^2 dr' d\Omega$$

$$= \sum_{\ell=0}^{\infty} \int_0^r \frac{r'^{\ell+2}}{r^{\ell+1}} \rho e(r') dr' \int_{\Omega} \left(\frac{4\pi}{2\ell+1}\right)^{\frac{1}{2}} Y_{\ell 0} d\Omega$$

$$= \sum_{\ell=0} \int_0^r dr' \; \frac{r'^{\ell+2}}{r^{\ell+1}} \; \rho_e(r') \; \frac{4\pi}{\sqrt{2\ell+1}} \; \delta_{\ell 0}\delta_{m0}$$

$$= \frac{4\pi}{r} \int_0^r r'^2 \rho_e(r') dr' \; .$$

For the second integral in (A-1), the same procedure can be applied

$$\iint_{\Omega\; r}^{\infty} \frac{1}{r'} \sum_{\ell=0}^{\infty} \left(\frac{r}{r'}\right)^{\ell} P_\ell(\cos\theta) \; \rho_e(r') r'^2 d\Omega \, dr'$$

$$= \int_r^{\infty} dr' \; \frac{r^{\ell}}{r'^{\ell-1}} \; \rho_e(r') \int_{\Omega} P_\ell(\cos\theta) d\Omega$$

$$= 4\pi \int_r^{\infty} r' \rho_e(r') dr' \; .$$

Thus the result is

$$V(r) = -4\pi Z e^2 \, \frac{1}{r} \left[\int_0^r r'^2 \rho_e(r') dr' + \int_r^{\infty} r' \rho_e(r') dr' \right] .$$

As an example, let us consider a uniform charge distribution

$$\rho_e(r') = \rho_e^0 \qquad (0 \leq r' \leq \rho) \; ,$$

$$\rho_e(r') = 0 \qquad (\rho < r') \; .$$

From the normalization condition, the relation between ρ and ρ_e^0 is found to be

$$4\pi \int_0^\infty \rho_e(r')r'^2 dr' = \rho_e^0 \, \frac{4\pi}{3} \rho^3 = 1$$

The potential V(r) due to the uniform charge distribution can be obtained, using eq.(1-3),

$$V(r) = -4\pi Ze^2\left[\frac{1}{r}\int_0^r \rho_e(r')r'^2 dr' + \int_r^\infty \rho_e(r')r'dr'\right]$$

$$= -4\pi Ze^2\left[\frac{1}{r}\int_0^r \rho_e^0 r'^2 dr' + \int_r^\rho \rho_e^0 r'dr' + \int_\rho^\infty 0r'dr'\right]$$

$$= -4\pi Ze^2\rho_e^0 \left[\frac{1}{r}\,\frac{r^3}{3} + \frac{1}{2}(\rho^2 - r^2)\right]$$

$$= -\rho_e^0 \frac{4\pi\rho^3}{3}\,\frac{Ze^2}{2\rho}\left(3 - \frac{r^2}{\rho^2}\right)$$

$$= -\frac{Ze^2}{2\rho}\left(3 - \frac{r^2}{\rho^2}\right)$$

$$= -\frac{\alpha Z}{2\rho}\left(3 - \frac{r^2}{\rho^2}\right) \qquad \text{for} \quad r < \rho$$

APPENDIX B

ANGULAR MOMENTUM AND DIRAC MATRICES

The gradient operator can be written in the form:

$$\vec{\nabla} = \hat{r}\frac{\partial}{\partial r} - i\frac{\hat{r}}{r}\times\vec{L} \qquad \text{where} \quad \vec{L} = \frac{1}{i}(\vec{r}\times\vec{\nabla})$$

$$= \frac{\vec{r}}{r}\frac{\partial}{\partial r} - i\frac{1}{r^2}\vec{r}\times\vec{L}.$$

To prove this equation, we write in terms of polar coordinates,

$$\vec{r} = x\hat{i} = y\hat{i} = z\hat{k} ,$$

$$x = r\sin\theta\cos\varphi , \quad y = r\sin\theta\sin\varphi , \quad z = r\cos\theta ,$$

$$\frac{\partial}{\partial r} = \sin\theta\cos\varphi\frac{\partial}{\partial x} + \sin\theta\sin\varphi\frac{\partial}{\partial y} + \cos\theta\frac{\partial}{\partial z} ,$$

hence

$$\frac{\vec{r}}{r}\frac{\partial}{\partial r} = \frac{x\hat{i} + y\hat{i} + z\hat{k}}{r}\left(\sin\theta\cos\varphi\frac{\partial}{\partial x} + \sin\theta\sin\varphi\frac{\partial}{\partial y} + \cos\theta\frac{\partial}{\partial z}\right) .$$

Let us consider the i-component only:

$$\hat{i} ; \quad \frac{x}{r}\left(\sin\theta\cos\varphi\frac{\partial}{\partial x} + \sin\theta\sin\varphi\frac{\partial}{\partial y} + \cos\theta\frac{\partial}{\partial z}\right)$$

$$= \frac{1}{r^2}\left(x^2\frac{\partial}{\partial x} + xy\frac{\partial}{\partial y} + xz\frac{\partial}{\partial z}\right)$$

hence

$$\left(\frac{\vec{r}}{r}\frac{\partial}{\partial r}\right)_i = \frac{1}{r^2}\left(x^2\frac{\partial}{\partial x} + xy\frac{\partial}{\partial y} + xz\frac{\partial}{\partial z}\right) . \qquad \text{(B-1)}$$

On the other hand, the i-component of $\vec{r} \times \vec{L}$ can be written in the form:

$$\left[-\frac{i}{r^2}\,\vec{r}\times\vec{L}\right]_i = -\frac{1}{r^2}\left(xy\frac{\partial}{\partial y} - y^2\frac{\partial}{\partial x} - z^2\frac{\partial}{\partial x} + xz\frac{\partial}{\partial z}\right) . \tag{B-2}$$

Therefore we have, combining eq.(B-1) and (B-2),

$$\begin{aligned}
\left[\frac{\vec{r}}{r}\,\frac{\partial}{\partial r} - \frac{i}{r^2}\,\vec{r}\times\vec{L}\right]_i &= \frac{1}{r^2}\Big(x^2\frac{\partial}{\partial x} + xy\frac{\partial}{\partial y} + xz\frac{\partial}{\partial z} \\
&\qquad - xy\frac{\partial}{\partial y} + y^2\frac{\partial}{\partial x} + z^2\frac{\partial}{\partial x} - xz\frac{\partial}{\partial z}\Big) \\
&= \frac{1}{r^2}\left(x^2 + y^2 + z^2\right)\frac{\partial}{\partial x} \\
&= \frac{\partial}{\partial x} \quad = (\vec{\nabla})_i \quad .
\end{aligned}$$

Similar equations can be obtained for $\hat{j}$ and $\hat{k}$ components. Thus we have

$$\vec{\nabla} = \frac{\vec{r}}{r}\,\frac{\partial}{\partial r} - \frac{i}{r^2}\,\vec{r}\times\vec{L}$$

$$\text{or} \qquad = \hat{r}\,\frac{\partial}{\partial r} - \frac{i}{r}\,\hat{r}\times\vec{L} \tag{B-3}$$

where $\hat{r} = \dfrac{\vec{r}}{r}$.

Next we prove the following expression

$$\vec{\alpha}\cdot\vec{A}\;\vec{\alpha}\cdot\vec{B} = \begin{pmatrix} \vec{A}\cdot\vec{B} & 0 \\ 0 & \vec{A}\cdot\vec{B} \end{pmatrix} + i\begin{pmatrix} \vec{\sigma}\cdot\vec{A}\times\vec{B} & 0 \\ 0 & \vec{\sigma}\cdot\vec{A}\times\vec{B} \end{pmatrix}$$

where

$$\vec{\alpha} = \begin{pmatrix} 0 & \vec{\sigma} \\ \vec{\sigma} & 0 \end{pmatrix}, \quad \beta = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix},$$

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

so that the left hand side of the above can be decomposed as

$$\begin{aligned}
(\vec{\alpha}\cdot\vec{A})(\vec{\alpha}\cdot\vec{B}) &= (\alpha_x A_x + \alpha_y A_y + \alpha_z A_z)(\alpha_x B_x + \alpha_y B_y + \alpha_z B_z) \\
&= \alpha_x^2 A_x B_x + \alpha_y^2 A_y B_y + \alpha_z^2 A_z B_z \\
&\quad + \alpha_x \alpha_y (A_x B_y - A_y B_x) \\
&\quad + \alpha_y \alpha_z (A_y B_z - A_z B_y) \\
&\quad + \alpha_z \alpha_x (A_z B_x - A_x B_z) \\
&= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} A_x B_x + \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} A_y B_y + \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} A_z B_z \\
&\quad + i\sigma_z (\vec{A}\times\vec{B})_z + i\sigma_x (\vec{A}\times\vec{B})_x + i\sigma_y (\vec{A}\times\vec{B})_y \\
&= \begin{pmatrix} \vec{A}\cdot\vec{B} & 0 \\ 0 & \vec{A}\cdot\vec{B} \end{pmatrix} + i \begin{pmatrix} \vec{\sigma}\cdot\vec{A}\times\vec{B} & 0 \\ 0 & \vec{\sigma}\cdot\vec{A}\times\vec{B} \end{pmatrix} .
\end{aligned} \tag{B-4}$$

Here we have used the fact that

$$\alpha_i \alpha_j = -\alpha_j \alpha_i = i\,\sigma_k .$$

Using eq. (B-4), we can easily show that

$$i\,\vec{\alpha}\cdot\hat{r}\times\vec{L} = \alpha_r \begin{pmatrix} \vec{\sigma}\cdot\vec{L} & 0 \\ 0 & \vec{\sigma}\cdot\vec{L} \end{pmatrix} .$$

Let $\vec{A} = \hat{r}$, $\vec{B} = \vec{L}$ in eq.(B-4) and using the expression $\vec{L} = \vec{r}\times\vec{p} = r\hat{r}\times\vec{p}$, we find

$$\vec{A}\cdot\vec{B} = \hat{r}\times\vec{L} = r\hat{r}\cdot(\hat{r}\times\vec{p}) = 0$$

Thus the first term of the right-hand side of eq. (B-4) is found to be zero. The rest is straightforward.

$$\vec{\alpha}\cdot\hat{r}\;\vec{\alpha}\cdot\vec{L} = i\begin{pmatrix}\vec{\sigma}\cdot\hat{r}\times\vec{L} & 0\\ 0 & \vec{\sigma}\cdot\hat{r}\times\vec{L}\end{pmatrix}$$

$$= i\begin{pmatrix}0 & I\\ I & 0\end{pmatrix}\begin{pmatrix}0 & \vec{\sigma}\cdot\hat{r}\times\vec{L}\\ \vec{\sigma}\cdot\hat{r}\times\vec{L} & 0\end{pmatrix}$$

$$= i\,\rho_3\vec{\alpha}\cdot\hat{r}\times\vec{L}$$

where

$$\rho_3 = \begin{pmatrix}0 & I\\ I & 0\end{pmatrix}, \quad \vec{\alpha} = \begin{pmatrix}0 & \vec{\sigma}\\ \vec{\sigma} & 0\end{pmatrix}$$

$$\rho_3\vec{\alpha} = \begin{pmatrix}0 & I\\ I & 0\end{pmatrix}\begin{pmatrix}0 & \vec{\sigma}\\ \vec{\sigma} & 0\end{pmatrix} = \begin{pmatrix}\vec{\sigma} & 0\\ 0 & \vec{\sigma}\end{pmatrix} = \vec{\sigma}$$

$$\rho_3\,\rho_3 = \begin{pmatrix}0 & I\\ I & 0\end{pmatrix}\begin{pmatrix}0 & I\\ I & 0\end{pmatrix} = I$$

$$\rho_3^{-1} = \rho_3, \quad [\rho_3, \vec{\alpha}] = [\vec{\alpha}, \rho_3]$$

$$i\,\rho_3\vec{\alpha}\cdot\hat{r}\times\vec{L} = \vec{\alpha}\cdot\hat{r}\;\vec{\alpha}\cdot\vec{L}$$

$$i\vec{\alpha}\cdot\hat{r}\times\vec{L} = \rho_3\vec{\alpha}\cdot\hat{r}\;\vec{\alpha}\cdot\vec{L}$$

$$= \vec{\alpha}\cdot\hat{r}\;\vec{\sigma}\cdot\vec{L}$$

$$= \alpha_r\begin{pmatrix}\vec{\sigma}\cdot\vec{L} & 0\\ 0 & \vec{\sigma}\cdot\vec{L}\end{pmatrix}, \qquad \text{(B-5)}$$

since ρ_3 and $\vec{\alpha}$ commute.

We now ready to prove eq.(1-6b). Using eqs.(B-3) and (B-5), we can write $\vec{\alpha}\cdot\vec{\nabla}$ as

$$\vec{\alpha}\cdot\vec{\nabla} = \vec{\alpha}\cdot\left(\hat{r}\,\frac{\partial}{\partial r} - i\,\frac{\hat{r}}{r}\times\vec{L}\right)$$

$$= \alpha_r\,\frac{\partial}{\partial r} - \frac{i}{r}\,\vec{\alpha}\cdot\hat{r}\times\vec{L}$$

$$= \alpha_r\,\frac{\partial}{\partial r} - \frac{1}{r}\,\alpha_r\begin{pmatrix}\vec{\sigma}\cdot\vec{L} & 0\\ 0 & \vec{\sigma}\cdot\vec{L}\end{pmatrix}$$

$$= \alpha_r\left(\frac{\partial}{\partial r} - \frac{1}{r}\,\vec{\sigma}\cdot\vec{L}\right) \qquad (1\text{-}6b)$$

APPENDIX C

EIGENVALUE OF $\sigma_r \chi_{\kappa\mu}$

$$\boxed{\sigma_r \chi_{\kappa\mu} = -\chi_{-\kappa\mu}}$$

Here we follow the derivation done by Rose.* We can write σ_r as

$$\sigma_r = \frac{1}{r}(\vec{r}\cdot\vec{\sigma}) = (\hat{r}\cdot\vec{\sigma}) = \left[\hat{r}^{(1)} \cdot \vec{\sigma}^{(1)}\right]^{(0)}, \tag{C-1}$$

since σ_r is a scalar product of two first-rank tensor. Using the Wigner-Eckart theorem and Racah coefficients W, the matrix element of two irreducible tensors of rank L can be written in the following form.

$$\langle j'm'j_1'j_2' | T_L(1)\cdot T_L(2) | jmj_1j_2\rangle$$

$$= \delta_{m'm}\,\delta_{j'j}\,(-)^{j_1'+j_2-j}\; W(j_1j_2j_1'j_2';\,jL)$$

$$\times\left[(2j_1'+1)(2j_2'+1)\right]^{\frac{1}{2}} \langle j_1'\|T_L(1)\|j_1\rangle\langle j_2'\|T_L(2)\|j_2\rangle \tag{C-2}$$

From eq.(C-2), we can readily see that

$$m' = m, \qquad j' = j.$$

If we write $\sigma_r \chi_{\kappa\mu}$ as

---

* See the reference 8)

$$\sigma_r \chi_{\kappa\mu} = \sum_{\kappa'\mu'} |\kappa'\mu'\rangle\langle\kappa'\mu'|\sigma_r|\kappa\mu\rangle$$

$$= \sum_{\kappa'\mu'} |\kappa'\mu'\rangle\langle\kappa'\mu'|\hat{r}^{(1)} \cdot \vec{\sigma}^{(1)}|\kappa\mu\rangle \qquad \text{(C-3)}$$

then, we have the same relation as is above, remembering that $j = \kappa - 1/2$, $\ell = \kappa$ for $\kappa > 0$, and $\ell = -(\kappa + 1)$ for $\kappa < 0$,

$$\mu' = \mu, \; |\kappa'| = |\kappa| \quad (j' = j) \qquad \text{(C-4)}$$

Using eq.(C-2), the right hand side of eq.(C-3) can be written as

$$\langle\kappa'\mu'|\hat{r}\cdot\vec{\sigma}|\kappa\mu\rangle = \langle|\kappa'|\,\mu'\,\ell'\,1/2|\,\hat{r}\cdot\vec{\sigma}\,|\,|\kappa|\,\mu\,\ell\,1/2\rangle$$

$$= \delta_{\mu\mu'}\,\delta_{|\kappa|,|\kappa'|}\,(-)^{\ell'+\frac{1}{2}-j}\,W(\ell\,1/2\,\ell'\,1/2;\,j1)$$

$$\times\sqrt{2(2\ell'+1)}\,\langle\ell'\|\hat{r}\|\ell\rangle\langle 1/2\|\vec{\sigma}\|1/2\rangle\,. \qquad \text{(C-5)}$$

Since σ_r has odd parity, we have $\ell' \neq \ell$. Thus $\kappa' = -\kappa$ so that $\ell' = \ell \pm 1$. The reduced matrix element of the angular momentum operator, in general, is

$$\langle j'\|J\|j\rangle = \{j(j+1)\}^{\frac{1}{2}}\,\delta_{j,j'}\,,$$

so that we obtain

$$\langle 1/2\|\vec{\sigma}\|1/2\rangle = 2\langle 1/2\|S\|1/2\rangle = \sqrt{3}\,. \qquad \text{(C-6)}$$

Using the reduced matrix element of the spherical harmonics between the eigenstates of the orbital angular momentum

$$\langle \ell_f \| Y_\ell \| \ell_i \rangle = C_{0\ 0\ 0}^{\ell_i\ \ell\ \ell_f} \left\{ \frac{(2\ell_i + 1)(2\ell + 1)}{4\pi} \right\}^{\frac{1}{2}},$$

we have

$$\langle \ell' \| \hat{r} \| \ell \rangle = - C_{0\ 0\ 0}^{\ell'\ 1\ \ell} \quad . \tag{C-7}$$

The relevant parts of the Racah coefficints and Clebsch-Gordan coefficents are as follows:

Racah coefficents

for $\ell = \ell' + 1$

$$(-)^{\ell'+\frac{1}{2}-j} \left\{ \frac{(j+\ell'+\frac{1}{2}+2)(j+\ell'-\frac{1}{2}+i)(\ell'+\frac{1}{2}-j+1)(j-\ell'+\frac{1}{2})}{4(2\ell'+1)(\ell'+1)(2\ell'+3)\frac{1}{2}\ \frac{3}{2}\ 2} \right\}^{\frac{1}{2}} \tag{C-8a}$$

for $\ell = \ell' - 1$

$$(-)^{\ell'+\frac{1}{2}-j-1} \left\{ \frac{(\ell'+\frac{1}{2}+j+1)(\ell'+\frac{1}{2}-j)(j+\ell'-\frac{1}{2})(j-\ell'+\frac{1}{2}+1)}{4(2\ell'+1)\ell'(2\ell'-1)\frac{1}{2}\ 2\ \frac{3}{2}} \right\}^{\frac{1}{2}} \tag{C-8b}$$

Clebsch-Gordan coefficents

for $\ell = \ell' + 1$

$$(+) \left\{ \frac{(\ell'+1)(\ell'+1)}{(2\ell'+1)(\ell'+1)} \right\}^{\frac{1}{2}} \tag{C-9a}$$

for $\ell = \ell' - 1$

$$(-) \left\{ \frac{\ell' \cdot \ell'}{\ell'(2\ell'+1)} \right\}^{\frac{1}{2}} \tag{C-9b}$$

Using eqs.(C-8a) to (C-9b), we can write eq.(C-5) as

$$\langle \kappa' \mu' | \hat{r}\cdot\vec{\sigma} | \kappa \mu \rangle = \langle \kappa' \mu' | \sigma_r | \kappa \mu \rangle$$

$$= -\delta_{\mu\mu'} \delta_{\kappa -\kappa'} \qquad \text{(C-10)}$$

Substituting eq.(C-10) into eq.(C-3) we finally arrive at the expression

$$\sigma_r \chi_{\kappa\mu} = \sum_{\kappa' \mu'} | \kappa' \mu' \rangle \, (-) \delta_{\mu\mu'} \delta_{\kappa -\kappa'}$$

$$= -\chi_{-\kappa\mu}$$

APPENDIX D

POWER-SERIES SOLUTIONS FOR THE "INSIDE" REGION

We shall discuss here the only relevant part of the second order differential equations which is essential for the power series expansion.

The standard form of the linear differential equation of the second order will be in the form

$$\frac{d^2u}{dz^2} + p(z)\frac{du}{dz} + q(z)u = 0$$

and it is assumed that there is a domain S in which both p(z) and q(z) are analytic except at a finite number of poles. Any point of S at which p(z) and q(z) are both analytic is said to be an ordinary point of the differential equation. If $z=z_0$ is not a ordinary point, but $(z-z_0)\ p(z)$, $(z-z_0)^2$ $q(z)$ are regular at z_0, z_0 is said to be a regular singular point of the differential equation. Now, if c be a regular singular point, the equation may be wtitten as

$$(z-c)^2\frac{d^2u}{dz^2} + (z-c)P(z-c)\frac{du}{dz} + Q(z-c)u = 0 \tag{D-1}$$

near a regular singular point, where P(z-c) and Q(z-c) are analytic at c. The Taylor expansion of these functions in powers of (z-c) are

$$P(z-c) = \sum_{n=0}^{\infty} p_n(z-c)^n$$

$$= p_0 + p_1(z-c) + p_2(z-c)^2 + \cdots\cdots \quad \text{(D-2a)}$$

$$Q(z-c) = q_0 + q_1(z-c) + q_2(z-c)^2 + \cdots\cdots \quad \text{(L-2b)}$$

where, p_0, p_1 -------, q_0, q_1 ------- are constants. These series converge in a domain Sc inside of a circle of radius r (center at c) where r is so small that c is the only singular point of the equation in Sc. Let us assume as a formal solution of the equation

$$u = (z-c)^{\alpha} \left[1 + \sum_{n=1}^{\infty} a_n(z-c)^n \right] \quad \text{(D-3)}$$

where , a_1, a_2 -------are constants to be determined. Substituting eq.(D-3) into (D-1), we get

$$\alpha(\alpha-1) + \sum_{n=1}^{\infty} (\alpha+n)(\alpha+n-1)\, a_n(z-c)^n$$

$$+ P(z-c)\left[\alpha + \sum_{n=1}^{\infty} a_n(z-c)^n(\alpha+n) \right]$$

$$+ Q(z-c)\left[1 + \sum_{n=1}^{\infty} a_n(z-c)^n \right] = 0 \quad \text{(D-4)}$$

Using eq.(D-2a) and (D-2b), we rewrite eq.(D-4),

$$\alpha(\alpha-1) + \sum_{n=1}^{\infty} (\alpha+n)(\alpha+n-1)\, a_n (z-c)^n$$

$$+ \left[p_0 + p_1(z-c) + p_2(z-c)^2 + \cdots\right] \left[\alpha + \sum_{n=1}^{\infty} a_n (z-c)^n (\alpha+n)\right]$$

$$+ \left[q_0 + q_1(z-c) + q_2(z-c)^2 + \cdots\right] \left[1 + \sum_{n=1}^{\infty} a_n (z-c)^n\right] = 0$$

Multiplying out and equating the coefficients of successive powers of (z-c) to zero, we obtain the following sequence of equations:

for n=0; $\alpha(\alpha-1) + p_0\alpha + q_0 = 0$ (D-5a)

for n=1; $(\alpha+1)\alpha a_1 + p_0 a_1(\alpha+1) + p_1\alpha + q_0 a_1 + q_1 = 0$ (D-5b)

From eq.(D-5a), we have

$$\alpha^2 + (p_0-1)\alpha + q_0 = 0 \tag{D-6}$$

This is called the indicial equation. From eq.(D-5b), we have

$$a_1 \left\{(\alpha+1)^2 + (p_0-1)(\alpha+1) + q_0\right\} + \alpha p_1 + q_1 = 0$$

The power-series expansion method is applied to the electron radial wave functions inside of a nucleus. Assuming the uniform charge distribution, we start with the first-order coupled differential equations. Suppressing the superscript (i) for the inside wave functions, we have

$$\frac{df_\kappa}{dr} = \frac{\kappa - 1}{r} f_\kappa - (W - 1 - V)\, g_\kappa(r) \tag{D-7}$$

$$\frac{dg_\kappa}{dr} = -\frac{\kappa + 1}{r} g_\kappa + (W + 1 - V)\, f_\kappa(r) \tag{D-8}$$

where $\quad V = -\frac{\alpha Z}{2\rho}\left(3 - \frac{r^2}{\rho^2}\right) = -\xi\left(3 - \frac{r^2}{\rho^2}\right)$.

Multiply (D-7) and (D-8) with r, we have,

$$r\frac{df_\kappa}{dr} = (\kappa - 1) f_\kappa - (W - 1 - V)\, r g_\kappa$$

$$r\frac{dg_\kappa}{dr} = -(\kappa + 1) g_\kappa + (W + 1 - V)\, r f_\kappa$$

$$\frac{d(rf)}{dr} = r\frac{df}{dr} + f \ .$$

Thus $\quad \frac{d(rf_\kappa)}{dr} = \kappa f_\kappa - (W - 1 - V)\, r g_\kappa$

$$\frac{d(rg_\kappa)}{dr} = -\kappa g_\kappa + (W + 1 - V)\, r f_\kappa \ .$$

Introducing

$$F_\kappa = r f_\kappa \ , \qquad G_\kappa = r g_\kappa$$

$$\frac{dF_\kappa}{dr} = \frac{\kappa}{r} F_\kappa - (W - 1 - V)\, G_\kappa \tag{D-9a}$$

$$\frac{dG_\kappa}{dr} = -\frac{\kappa}{r} G_\kappa + (W + 1 - V) F_\kappa \,. \tag{D-9b}$$

We rewrite eq.(D-9a) and (D-9b) in the following way:

$$\frac{dF_\kappa}{dr} - \frac{\kappa}{r} F_\kappa = -\frac{G_\kappa}{P_1}$$

$$\frac{dG_\kappa}{dr} + \frac{\kappa}{r} G_\kappa = \frac{F_\kappa}{P_2} \,.$$

where

$$P_1^{-1} = W - 1 + \xi(3 - \frac{r^2}{\rho^2})$$

$$P_2^{-1} = W + 1 + \xi(3 - \frac{r^2}{\rho^2})$$

or

$$P_1 \left(\frac{dF_\kappa}{dr} - \frac{\kappa}{r} F_\kappa\right) = -G_\kappa \tag{D-10a}$$

$$P_2 \left(\frac{dG_\kappa}{dr} + \frac{\kappa}{r} G_\kappa\right) = F_\kappa \,. \tag{D-10b}$$

From eq.(D-10a)

$$\frac{\kappa}{r} P_1\left(\frac{dF_\kappa}{dr} - \frac{\kappa}{r} F_\kappa\right) = -\frac{\kappa}{r} G_\kappa \,. \tag{D-11a}$$

Differentiating both sides of eq.(D-10a)

$$\frac{d}{dr}\left[P_1\left(\frac{dF_\kappa}{dr} - \frac{\kappa}{r} F_\kappa\right)\right] = -\frac{dG_\kappa}{dr} \,. \tag{D-11b}$$

Adding eqs.(D-11a) and (D-11b), it is found that

$$\frac{d}{dr}\left[P_1\left(\frac{dF_\kappa}{dr} - \frac{\kappa}{r}F_\kappa\right)\right] + \frac{\kappa}{r}P_1\left(\frac{dF_\kappa}{dr} - \frac{\kappa}{r}F_\kappa\right) = -\frac{dG_\kappa}{dr} - \frac{\kappa}{r}G_\kappa \,.$$

Using eq.(D-10b), we have

$$\frac{d}{dr}\left[P_1\left(\frac{dF_\kappa}{dr} - \frac{\kappa}{r}F_\kappa\right)\right] + \frac{\kappa}{r}P_1\left(\frac{dF_\kappa}{dr} - \frac{\kappa}{r}F_\kappa\right) = -\frac{F_\kappa}{P_2}$$

$$P_1\left(\frac{d^2F_\kappa}{dr} - \frac{\kappa}{r}\frac{dF_\kappa}{dr} + \frac{\kappa}{r^2}F_\kappa\right) + \frac{dP_1}{dr}\left(\frac{dF_\kappa}{dr} - \frac{\kappa}{r}F_\kappa\right)$$

$$+ P_1\left(\frac{\kappa}{r}\frac{dF_\kappa}{dr} - \frac{\kappa^2}{r^2}F_\kappa\right) = -\frac{F_\kappa}{P_2} \quad .$$

Thus

$$\frac{d^2F_\kappa}{dr^2} + \frac{\kappa-\kappa^2}{r^2}F_\kappa + \frac{\frac{dP_1}{dr}}{P_1}\left(\frac{dF_\kappa}{dr} - \frac{\kappa}{r}F_\kappa\right) + \frac{F_\kappa}{P_1P_2} = 0 \quad .$$

Similarly the second order differential equation for G_κ is

$$\frac{d^2G_\kappa}{dr^2} - \frac{\kappa+\kappa^2}{r^2}G_\kappa + \frac{\frac{dP_1}{dr}}{P_2}\left(\frac{dG_\kappa}{dr} + \frac{\kappa}{r}G_\kappa\right) + \frac{G_\kappa}{P_1P_2} = 0 \quad .$$

Finally we have the uncoupled second order differential equations.

$$\frac{d^2F_\kappa}{dr^2} + \frac{d}{dr}(\ln P_1)\frac{dF_\kappa}{dr} + \left[\frac{\kappa(1-\kappa)}{r^2} - \frac{\kappa}{r}\frac{d}{dr}(\ln P_1) + (P_1P_2)^{-1}\right]F_\kappa = 0 \tag{D-12a}$$

$$\frac{d^2G_\kappa}{dr^2} + \frac{d}{dr}(\ln P_2)\frac{dG_\kappa}{dr} + \left[-\frac{\kappa(\kappa+1)}{r^2} + \frac{\kappa}{r}\frac{d}{dr}(\ln P_2) + (P_1P_2)^{-1}\right]G_\kappa = 0 \tag{D-12b}$$

Using the definition of P_1 and P_2 , we find

$$(P_1P_2)^{-1} = \left\{ W + \xi(3 - \frac{r^2}{\rho^2}) \right\}^2 - 1$$

$$\frac{P_1'}{P_1} = (\ln P_1)' = \frac{d}{dr}\left[-\ln\left\{ W - 1 + \xi(3 - \frac{r^2}{\rho^2})\right\}\right]$$

$$= \frac{-2r}{r^2 - \frac{(W-1)\rho^2 + 3\xi\rho^2}{\xi}} .$$

Let r_1 denote the solution of $r^2 - \frac{(W-1)\rho^2 + 3\xi\rho^2}{\xi} = 0$

$$\frac{P_1'}{P_1} = \frac{-2r}{(r+r_1)(r-r_1)} = -\left(\frac{1}{r-r_1} + \frac{1}{r+r_1}\right)$$

$$r_1 = \pm\rho\sqrt{3 + \frac{W-1}{\xi}} . \qquad \text{(D-13a)}$$

In a similar way, we can also show that $(\ln P_2)'$ satisfies

$$(\ln P_2)' = \frac{-2r}{(r+r_2)(r-r_2)} = -\left(\frac{1}{r-r_2} + \frac{1}{r+r_2}\right)$$

where

$$r_2 = \pm\rho\sqrt{3 + \frac{W+1}{\xi}} . \qquad \text{(D-13b)}$$

Then eqs.(D-12a) and (D-12b) can be rewritten as

$$\frac{d^2F_\kappa}{dr^2} - \left(\frac{1}{r-r_1} + \frac{1}{r+r_1}\right)\frac{dF_\kappa}{dr} + \left[-\frac{\kappa(\kappa-1)}{r^2} + \frac{\kappa}{r}\left(\frac{1}{r-r_2} + \frac{1}{r-r_2}\right) + \frac{\xi^2}{\rho^4}(r-r_1)(r+r_1)(r-r_2)(r-r_2)\right] F_\kappa = 0 \qquad \text{(D-14a)}$$

$$\frac{d^2G_\kappa}{dr^2} - \left(\frac{1}{r-r_2} + \frac{1}{r+r_2}\right)\frac{dG_\kappa}{dr} + \left[-\frac{\kappa(\kappa+1)}{r^2} - \frac{\kappa}{r}\left(\frac{1}{r-r_2} + \frac{1}{r-r_2}\right) + \frac{\xi^2}{\rho^4}(r-r_1)(r+r_1)(r-r_2)(r+r_2)\right] G_\kappa = 0 \quad . \qquad \text{(D-14b)}$$

Using the indicial equation (D-6), we have, from (D-14a) and (D-14b)

$$\alpha^2 - \alpha - \kappa(\kappa-1) = 0 \qquad \text{for} \quad F_\kappa \ ,$$

$$\alpha^2 - \alpha - \kappa(\kappa+1) = 0 \qquad \text{for} \quad G_\kappa \ .$$

Thus, $\alpha = \kappa$ or $1-\kappa$ for F_κ and $\alpha = -\kappa$ or $\kappa+1$ for G_κ. Suppose if κ is negative, we obtain

$$\alpha = 1 - \kappa = 1 + k \qquad \text{for} \quad F_\kappa \ , \qquad \text{(D-15a)}$$

$$\alpha = -\kappa = k \qquad \text{for} \quad G_\kappa \qquad \text{(D-15b)}$$

where $k = |\kappa|$.

Combining eqs.(D-3),(D-15a) and (D-15b), we arrive at the final expression for $\kappa < 0$.

$$F_\kappa = r^{1+k} \sum_{n=0}^{\infty} A_n r^n \overset{*}{=} r^{1+k} \sum_{n=0}^{\infty} a_n \qquad \text{(D-16a)}$$

$$G_\kappa = r^k \sum_{n=0}^{\infty} B_n r^n = r^k \sum_{n=0}^{\infty} b_n \qquad \text{(D-16b)}$$

---

* $F(r) = r^\alpha \sum_{n=0}^{\infty} A_n r^n = A_0 r^\alpha \left(1 + \sum_{n=1}^{\infty} A_n' r^n\right)$ where $A_n' = \dfrac{A_n}{A_0}$

Since the coefficients A_n and B_n of odd powers of r may be found to be zero, we use the expansion of the form

$$F^{(i)}(r) = r^{k+1} \sum_{n=0}^{\infty} A_n r^{2n} = r^{k+1} \sum_{n=0}^{\infty} a_n \quad , \qquad \text{(D-17a)}$$

$$G^{(i)}(r) = r^{k} \sum_{n=0}^{\infty} B_n r^{2n} = r^{k} \sum_{n=0}^{\infty} b_n \quad . \qquad \text{(D-17b)}$$

Substituting eqs.(D-17) into (D-9) and equating the coefficients of the same order of powers of r, we obtain

$$r^0 : \quad (k+1)A_0 = \varkappa A_0 - (W-1+3\xi)B_0 \qquad \text{(D-18a)}$$

$$k\,B_0 = -\varkappa\, B_0 \quad . \qquad \text{(D-18b)}$$

$$r^{2n}: \quad (2k+2n+1)A_n = -(W-1+3\xi)B_n + \frac{\xi}{\rho^2} B_{n-1} \qquad \text{(D-18c)}$$

$$2n\, B_n = (W+1+3\xi)A_{n-1} - \frac{\xi}{\rho^2} A_{n-2} \quad . \qquad \text{(D-18d)}$$

From eq.(D-18b), $\varkappa$ must be negative to be consistent with our assumption $\varkappa < 0$. Using eqs.(D-17) and (D-18), we obtain the recurrence relations for a_n and b_n at $r = \rho$.

$$a_n = \frac{1}{2k+2n+1} \left[-(W-1+3\xi)b_n + \xi b_{n-1} \right] \qquad \text{(D-19a)}$$

$$b_n = \frac{\rho^2}{2n} \left[(W+1+3\xi)a_{n-1} - \xi a_{n-2} \right] \qquad \text{(D-19b)}$$

APPENDIX E

GUDERMANNIAN FUNCTION AND HYPERBOLIC FUNCTION

Let θ denote the gudermannian of x, written as

$$\theta = gd(x)$$

$$= 2\tan^{-1}(e^{x}) - \pi/2$$

and its inverse function as

$$x = gd^{-1}\theta$$

$$= \ln\{\tan(\theta/2 + \pi/4)\} \quad \text{(E-1)}$$

Relations with hyperbolic functions are as follows:

$$\sinh x = \tan gd(x) ,$$

$$\tan x = \sinh (gd^{-1}x) ,$$

$$\tanh(x/2) = \tan\frac{gd\ x}{2} ,$$

and $$\sinh x = (e^{x} - e^{-x})/2 ;$$

$$\cosh x = (e^{x} + e^{-x})/2 .$$

Let us define ζ as

$$2\zeta = gd((\rho - r)/a\rho),$$

then $$gd^{-1}\ 2\zeta = (\rho - r)/a\rho .$$

Now using eq.(E-1), it is found that

$$\frac{1}{2}gd^{-1}4\zeta = \frac{1}{2}gd^{-1}2(2\zeta) = \frac{1}{2}\ln\left\{\tan(\pi/4 + 2\zeta)\right\} , \tag{E-2}$$

and that

$$e^{-\frac{1}{2}gd^{-1}(2gd\frac{\rho - r}{a\rho})} = e^{-\frac{1}{2}gd^{-1}(2\cdot 2\zeta)}$$

$$= e^{-\frac{1}{2}\ln\frac{1+\tan 2\zeta}{1-\tan 2\zeta}} = e^{-\tanh^{-1}(\tan\, gd\,\frac{\rho - r}{a\rho})}$$

$$= e^{-\tanh^{-1}(\sinh\frac{\rho - r}{a\rho})} .$$

Now from this equation, we have

$$e^{-gd^{-1}(2gd\frac{\rho - r}{a\rho})} = e^{-gd^{-1}(4\zeta)}$$

$$= e^{-2\tanh^{-1}(\sinh\frac{\rho - r}{a\rho})} . \tag{E-3}$$

Using eq.(E-3), we arrive at the expression

$$\left[1 + e^{-gd^{-1}4\zeta}\right]^{-1} = \frac{1}{1 + e^{-2\tanh^{-1}(\sinh\frac{\rho - r}{a\rho})}}$$

$$= \frac{1}{2}\left[1 + \sinh\frac{\rho - r}{a\rho}\right] . \tag{E-4}$$

APPENDIX F

NORMALIZATION CONDITION

OF A DIFFUSED CHARGE DISTRIBUTION

Using eq.(3-5), we discuss what conditions can be imposed on a and ρ_e^0.

$$\int_0^\infty \rho_e(r')r'^2dr'$$

$$= \int_0^{\rho-t} \rho_e^0 r'^2dr' + \int_{\rho-t}^{\rho+t} \frac{\rho_e^0}{2}(1 + \sinh\frac{\rho - r'}{a\rho})r'^2dr'$$

$$= \frac{\rho_e^0}{2}\left[\int_0^{\rho-t} 2r'^2dr' + \int_{\rho-t}^{\rho+t}(1 + \sinh\frac{\rho-r'}{a\rho})\, r'^2dr'\right]$$

$$= \frac{\rho_e^0}{2}\left[\frac{2}{3}(\rho-t)^3 + \frac{1}{3}\left\{(\rho+t)^3 - (\rho-t)^3\right\} + \int_{\rho-t}^{\rho+t}\sinh\frac{\rho-r'}{a\rho}r'^2dr'\right]$$

$$= \frac{\rho_e^0}{2}\left[\frac{1}{3}\left\{(\rho+t)^3 + (\rho-t)^3\right\} + \int_{\rho-t}^{\rho+t}\sinh\frac{\rho-r'}{a\rho}r'^2dr'\right]. \qquad \text{(F-1)}$$

Hence at $r' = \rho - t$, the following equation must be satisfied:

$$\rho_e^0 = \frac{\rho_e^0}{2}(1 + \sinh\frac{\rho-r'}{a\rho}),$$

so that we find

$$\sinh\frac{\rho-r'}{a\rho} = 1,$$

or $\quad r' = \rho(1 - a\sinh^{-1}(1))$.

Similarly at $r' = \rho + t$, we have

$$0 = \frac{\rho_e^0}{2}\left(1 + \sinh\frac{\rho - r'}{a\rho}\right)$$

so that

$$r' = \rho(1 - a\sinh^{-1}(-1)) .$$

Therefore it is found that the thickness t is related <u>a</u> by

$$t = a\rho\sinh^{-1}(1) . \tag{F-2}$$

To perform the integration over r', we define a new variable x as $x = (\rho - r')/a\rho$ so that we have $x = -\sinh^{-1}(1)$ at the upper boundary and $x = \sinh^{-1}(1)$ at the lower boundary .

$$\int_{\rho-t}^{\rho+t} \sinh\frac{\rho - r'}{a\rho} r'^2 dr' = -a\rho^3 \int_{\sinh^{-1}(1)}^{-\sinh^{-1}(1)} (1 - ax)^2 \sinh x \, dx$$

$$= a\rho^3 \Big[\cosh x - 2a(x\cosh x - \sinh x) + a^2\{(x^2+2)\cosh x - 2x\sinh x\}\Big|_{-\sinh^{-1}(1)}^{\sin^{-1}(1)} .$$

Since cosh x and sinh x are even and odd function, respectively, we can now wtite

$$\cosh x\Big|_{-\sinh^{-1}(1)}^{\sinh^{-1}(1)} = 0 , \quad (x^2+2)\cosh x - 2x\sinh x\Big|_{-\sinh^{-1}(1)}^{\sinh^{-1}(1)} = 0$$

and

$$x \cosh x - \sinh x \Big|_{-\sinh^{-1}(1)}^{\sin^{-1}(1)}$$

$$= \sinh^{-1}(1) \cosh \cosh^{-1}(\sqrt{2}) - 1 + \sin^{-1}(1) \cosh(\cosh^{-1}\sqrt{2}\ ) - 1$$

$$= 2\sqrt{2} \sinh^{-1}(1) - 2 \ .$$

Here we have used

$$\sinh^{-1}x = \pm \cosh^{-1}\sqrt{x^2+1} \quad \begin{pmatrix} + \text{ for } & x > 0 \\ - \text{ for } & x < 0 \end{pmatrix} .$$

It is therefore found that

$$\int_{\rho-t}^{\rho+t} \sinh\frac{\rho - r'}{a\rho}\, r'^2 dr' = -4a^2\rho^3(\sqrt{2}\sinh^{-1}(1) - 1). \quad \text{(F-3)}$$

Inserting eq.(F-3) into eq.(F-1) yields

$$\int \rho_e(r')r'^2dr'$$

$$= \frac{\rho_e^0}{2}\left[\frac{1}{3}\{(\rho+t)^3 + (\rho-t)^3\} - 4a^2\rho^3(\sqrt{2}\sinh^{-1}(1) - 1)\right]$$

$$= \rho_e^0\, \rho^3 \left[\frac{1}{3} + a^2\left\{\sinh^{-1}(1) - \sqrt{2}\right\}^2\right] .$$

Thus, from the normalization condition, we have

$$4\pi\rho_e^0\, \rho^3 \left[\frac{1}{3} + a^2\left\{\sinh^{-1}(1) - \sqrt{2}\right\}^2\right] = 1 .$$

Hence

$$\rho_e^0 = \frac{1}{\frac{4\pi}{3}\rho^3} \; \frac{1}{1 + 3a^2(\sinh^{-1}(1) - \sqrt{2})^2} \quad . \qquad \text{(F-4)}$$

It might be helpful to give general formula

$$\int x^n \sinh x \, dx = x^n \cosh x - n \int x^{n-1}(\cosh x) dx, \text{ and}$$

for n = 1

$$\int x \sinh x \, dx = x \cosh x - \int \cosh x \, dx$$

$$= x \cosh x - \sinh x \quad ,$$

for n = 2

$$\int x^2 \sinh x \, dx = x^2 \cosh x - 2 \int x \cosh x \, dx$$

$$= x^2 \cosh x - 2(x \sinh x - \int \sinh x \, dx)$$

$$= x^2 \cosh x - 2x \cdot \sinh x + 2 \cdot \cosh x$$

$$= (x^2+2) \cosh x - 2x \cdot \sinh x \; .$$

APPENDIX G

POTENTIAL ASSOCIATED WITH DIFFUSED CHARGE DISTRIBUTION

We now calcutate the potential in each region. Let us define $I^{(0)}$ and $I^{(i)}$ as

$$-\frac{V(r)}{4\pi Ze^2} = \begin{cases} I^{(0)} & \text{for } 0 \leq r \leq \rho - t \\ I^{(i)} & \text{for } \rho - t \leq r \leq \rho + t \end{cases},$$

remembering that the charge density distribution is given by

$$\rho_e(r) = \rho_e^0 \qquad (0 \leq r < \rho - t)$$

$$\rho_e(r) = \frac{\rho_e^0}{2}\left(1 + \sinh\frac{\rho - r}{a\rho}\right) \qquad (\rho - t \leq r \leq \rho + t)$$

$$\rho_e(r) = 0 \qquad (\rho + t < r) .$$

We can apply the same method used in Appendix F to determine the form of the potential.

$$I^{(0)} = \frac{1}{r}\int_0^r \rho_e^0 \, r'^2 dr' + \int_r^{\rho - t} \rho_e^0 \, r' dr'$$

$$+ \int_{\rho - t}^{\rho + t} \frac{\rho_e^0}{2}\left(1 + \sinh\frac{\rho - r'}{a\rho}\right) r' dr'$$

$$= \rho_e^0 \frac{r^3}{3r} + \rho_e^0 \times \frac{(\rho - t)^2 - r^2}{2} + \frac{\rho_e^0}{2}\,\frac{1}{2}\left\{(\rho+t)^2 - (\rho-t)^2\right\}$$

$$+ \frac{\rho_e^0}{2}\int_{\rho-t}^{\rho+t} r' \sinh\frac{\rho - r'}{a\,\rho}\, dr' \,. \qquad \text{(G-1)}$$

And

$$\int_{\rho-t}^{\rho+t} r' \sinh\frac{\rho - r'}{a\,\rho}\, dr'$$

$$= -a\rho^2 \int_{\sinh^{-1}(1)}^{-\sinh^{-1}(1)} (\sinh x - ax \sinh x)\,dx$$

$$= -a\rho^2 \Big[\cosh x - a(x \cosh x - \sinh x)\Big]_{\sinh^{-1}(1)}^{-\sinh^{-1}(1)}$$

$$= -2a^2\rho^2 \left\{\sqrt{2}\,\sinh^{-1}(1) - 1\right\}. \qquad \text{(G-2)}$$

Thus substituting eq.(G-2) into eq.(G-1), we find

$$I^{(0)} = \rho_e^0 \frac{r^2}{3} + \rho_e^0 \frac{(\rho-t)^2 - r^2}{2} + \frac{\rho_e^0}{4}\left\{(\rho+t)^2 - (\rho-t)^2\right\}$$

$$+ \frac{\rho_e^0}{2}(-2a^2\rho^2)(\sqrt{2}\,\sinh^{-1}(1) - 1)$$

$$= \frac{\rho_e^0\,\rho^2}{2}\left[\frac{1}{3}\,(3 - \frac{r^2}{\rho^2}) + a^2(\sinh^{-1}(1) - \sqrt{2})^2\right].$$

With the help of the normalization condition, the potential V(r) for the innermost region can be written as

$$V(r) = -4\pi Ze^2 I^{(0)}.$$

$$= -\frac{\alpha Z}{2\rho} \cdot \frac{(3 - \frac{r^2}{\rho^2}) + 3a^2(\sinh^{-1}(1) - \sqrt{2}\ )^2}{1 + 3a^2(\sinh^{-1}(1) - \sqrt{2}\ )^2} \quad . \tag{G-3}$$

Note that if a approaches zero, or equivalently the thickness approaches zero, we have

$$V(r) = -\frac{\alpha Z}{2\rho}(3 - \frac{r^2}{\rho^2}\ ) \quad .$$

which is the potential derived from the uniform charge distribution.

Now we calculate the integral $I^{(i)}$. $I^{(i)}$ can be obtained in the following way.

$$I^{(i)} = \frac{1}{r}\int_0^{\rho-t} \rho_e^0\ r'^2 dr' + \frac{1}{r}\int_{\rho-t}^{r} \frac{\rho_e^0}{2}(1 + \sinh\frac{\rho-r'}{a\rho})r'^2 dr'$$

$$+\int_r^{\rho+t} \frac{\rho_e^0}{2}(1 + \sinh\frac{\rho-r'}{a\rho})r' dr'$$

$$= \rho_e^0\,\frac{(\rho-t)^3}{3r} + \frac{\rho_e^0}{2r}\left[\frac{1}{3}\left\{r^3 - (\rho-t)^3\right\} + \int_{\rho-t}^{r} r'^2 \sinh\frac{\rho-r'}{a\rho}dr'\right]$$

$$+ \frac{\rho_e^0}{2}\left[\frac{1}{2}\left\{(\rho+t)^2 - r^2\right\} + \int_r^{\rho+t} r' \sinh\frac{\rho-r'}{a\rho}dr'\right] \quad . \tag{G-4}$$

And

$$\int_{\rho - t}^{r} r'^2 \sinh\frac{\rho - r'}{a\rho}\, dr'$$

$$= -a\rho^3 \left[\cosh x - 2a(x \cosh x - \sinh x) + a^2\{x^2 \cosh x - 2(x \sinh x - \cosh x)\} \right]_{\sinh^{-1}(1)}^{\frac{\rho - r}{a\rho}}$$

$$= -a\rho^3 \Bigg[\cosh\frac{\rho - r}{a\rho} - 2a\left(\frac{\rho - r}{a\rho} \cosh\frac{\rho - r}{a\rho} - \sinh\frac{\rho - r}{a\rho}\right)$$

$$+ a^2 \left\{ \left(\left(\frac{\rho - r}{a\rho}\right)^2 + 2\right) \cosh\frac{\rho - r}{a\rho} - 2\frac{\rho - r}{a\rho}\sinh\frac{\rho - r}{a\rho} \right\}$$

$$- \sqrt{2} + 2a(\sqrt{2}\sinh^{-1}(1) - 1) - a^2 \{ \sqrt{2}(\sinh^{-1}(1))^2$$

$$+ 2\sqrt{2} - 2\sinh^{-1}(1) \} \Bigg]$$

$$= -a\rho^3 \Bigg[\cosh\frac{\rho - r}{a\rho} \left\{ 1 - 2a\frac{\rho - r}{a\rho} + a^2\left(\left(\frac{\rho - r}{a\rho}\right)^2 + \cancel{2}\right) \right\}$$

$$+ 2a \sinh\frac{\rho - r}{a\rho}\left(1 - a\frac{\rho - r}{a\rho}\right) - \sqrt{2}\left(2a^2 + (a \sinh^{-1}(1) - 1)^2\right)$$

$$+ 2a(1 - a \sinh^{-1}(1)) \Bigg] \qquad \text{(G-5)}$$

$$\int_{r}^{\rho + t} r' \sinh\frac{\rho - r'}{a\rho}\, dr'$$

$$= - a\rho^2 \left[\cosh x - a(x \cosh x - \sinh x) \right]_{\frac{\rho - r}{a\rho}}^{-\sinh^{-1}(1)}$$

$$= - a\rho^2 \left[\sqrt{2}\,(1 + a \sinh^{-1}(1)) - a - (1 - a\frac{\rho - r}{a\rho})\cosh\frac{\rho - r}{a\rho} \right.$$

$$\left. - a \sinh\frac{\rho - r}{a\rho} \right] . \tag{G-6}$$

It is therefore found, substituting eqs.(G-5) and (G-6) into eq.(G-4), that the potential V(r) can be expressed as

$$V(r) = \frac{\alpha Z}{4\rho} \left\{ 1 + 3a^2 (\sqrt{2} - \sinh^{-1}(1))^2 \right\}^{-1}$$

$$\times \left[3 \left\{ (1 + \frac{t}{\rho})^2 - 2a((\sqrt{2} + \frac{t}{\rho}) - a) - 2a^2 \sinh\frac{\rho - r}{a\rho} \right\} - \frac{r^2}{\rho^2} \right.$$

$$+ \frac{2}{r} \left\{ \rho(1 - \frac{t}{\rho})^3 + 3a\rho \left(\sqrt{2}\,(2a^2 + (1 - \frac{t}{\rho})^2) + 2a(1 - \frac{t}{\rho}) \right) \right.$$

$$\left. \left. - 6a^3\rho \cosh\frac{\rho - r}{a\rho} \right\} \right] . \tag{G-7}$$

Now considering $\sinh \{ (\rho - r)/a\rho \} = 1$ and

$$\cosh \{ (\rho - r)/a\rho \} = \sqrt{2}$$

at the inner boundary $r = \rho - t$, we find that

$$\overset{in}{V(r)} = \frac{-1}{2} \xi' \left\{ 3\overset{in}{C_A} - \frac{r^2}{\rho^2} + \frac{2\rho}{r} C_B \right\} \tag{G-8}$$

where

$$\xi' = \frac{\alpha Z}{2\rho}\left\{ 1 + 3a^2(\sqrt{2} - \sinh^{-1}(1))^2 \right\}^{-1} ,$$

$$C_A^{in} = (1 + \frac{t}{\rho})^2 - 2a\sqrt{2}\,(1 + \frac{t}{\rho}) \quad ,$$

$$C_B = (1 - \frac{t}{\rho})^3 + 3a\left\{ \sqrt{2}\,(1 - \frac{t}{\rho})^2 + 2a(1 - \frac{t}{\rho}) \right\} .$$

Similarly at the outer boundary, i.e., $r = \rho + t$,

$$\sinh\left\{(\rho - r)/a\rho\right\} = -1 \text{ and } \cosh\left\{(\rho - r)/a\rho\right\} = \sqrt{2}$$

so that

$$V^{out}(r) = \frac{-1}{2}\xi'\left\{ 3C_A^{out} - \frac{r^2}{\rho^2} + \frac{2\rho}{r} C_B \right\} \qquad \text{(G-9)}$$

where

$$C_A^{out} = (1 + \frac{t}{\rho})^2 - 2a\sqrt{2}\,(1 + \frac{t}{\rho}) + 4a^2$$

APPENDIX H

POWER SERIES SOLUTIONS

FOR THE "INTERMEDIATE" REGION

As was carried out in the Appendix D for the uniform charge distribution, we start with the differential equation

$$\frac{dF}{dr} = \frac{\kappa}{r} F - (W - 1 - V)G^{*} , \tag{H-1}$$

$$\frac{dG}{dr} = - \frac{\kappa}{r} G + (W + 1 - V)F , \tag{H-2}$$

where $$V = - \frac{\xi}{2} (3 - \frac{r^2}{\rho^2} + 2\frac{\rho}{r}) . \tag{H-3}$$

Introducing a new variable x defined by

$$x = \frac{r}{\rho} ,$$

we then have the differential equation in the folloing form.

$$P_1 (\frac{dF}{dx} - \frac{\kappa}{x} F) = - G , \tag{H-4}$$

$$P_2 (\frac{dG}{dx} + \frac{\kappa}{x} G) = F \tag{H-5}$$

where $$P_1^{-1} = \rho \{W - 1 + \frac{\xi}{2}(3 - x^2 + \frac{2}{x})\} ,$$

$$P_2^{-1} = \rho \{W + 1 + \frac{\xi}{2}(3 - x^2 + \frac{2}{x})\} .$$

---

* We surpress the superscript (i) for F and G throughout this appendix.

From eqs. (H-4) and (H-5), we can readily obtain the second order differential equations for F and G.

$$F'' + \frac{P_1'}{P_1} F' + \left[- \frac{\kappa(\kappa-1)}{x^2} - \frac{P_1'}{P_1} \frac{\kappa}{x} + \frac{1}{P_1 P_2} \right] F = 0 , \tag{H-6}$$

$$G'' + \frac{P_2'}{P_2} G' + \left[- \frac{\kappa(\kappa+1)}{x^2} + \frac{P_2'}{P_2} \frac{\kappa}{x} + \frac{1}{P_1 P_2} \right] G = 0 , \tag{H-7}$$

where prime means the differentiation with respect to x. Now P_1' / P_1 can be written as

$$\frac{P_1'}{P_1} = (\ln P_1)' = \frac{1}{x} - \frac{3x^2 - (3 + 2\frac{W-1}{\xi})}{x^3 - (3 + 2\frac{W-1}{\xi})x - 2}$$

Let x_1^- , x_2^- , x_3^- be the solution of the equation

$$x^3 - (3 + 2\frac{W-1}{\xi})x - 2 = 0 .$$

We then have the expression

$$\frac{P_1'}{P_1} = \frac{1}{x} - \frac{1}{x-x_1^-} + \frac{1}{x-x_2^-} + \frac{1}{x-x_3^-}$$

$$= \frac{1}{x} - \sum_i \frac{1}{x-x_i^-} . \tag{H-8a}$$

In the similar manner, $\frac{P_2'}{P_2}$ cam be written as

$$\frac{P_2'}{P_2} = \frac{1}{x} - \sum_i \frac{1}{x-x_i^+} \tag{H-8b}$$

where x_i^+ are the solution of the equation

$$x^3 - (3 + 2\frac{W+1}{\xi})x - 2 = 0 \; .$$

Using the solution x_i^- and x_i^+, we obtain

$$\frac{1}{P_1P_2} = \left(\frac{\rho\xi}{2x}\right)^2 \left[x^3 - \{3 + \frac{2}{\xi}(W-1)\}x - 2\right]\left[x^3 - \{3 + \frac{2}{\xi}(W+1)\}x - 2\right]$$

$$= \left(\frac{\rho\xi}{2x}\right)^2 \prod_i (x-x_i^-) \prod_i (x-x_i^+) \tag{H-9}$$

$$= \left(\frac{\rho\xi}{2x}\right)^2 \prod_i (x-x_i^{\pm}) \; .$$

Substituting eqs. (H-8a), (H-8b) and (H-9) into (H-6) and (H-7) we therefore have

$$\frac{d^2F}{dx^2} + \left(\frac{1}{x} - \sum_i \frac{1}{x-x_i^-}\right)\frac{dF}{dx} + \left\{- \frac{\kappa(\kappa-1)}{x^2} - \left(\frac{1}{x} - \sum \frac{1}{x-x_i^-}\right)\frac{\kappa}{x}\right.$$

$$\left. + \left(\frac{\rho\xi}{2x}\right)^2 \prod_i (x-x_i^-) \prod_i (x-x_i^+)\right\} F = 0 \tag{H-10}$$

$$\frac{d^2G}{dx^2} + \left(\frac{1}{x} - \sum_i \frac{1}{x-x_i^+}\right)\frac{dG}{dx} + \left\{- \frac{\kappa(\kappa+1)}{x^2} + \left(\frac{1}{x} - \sum \frac{1}{x-x_i^+}\right)\frac{\kappa}{x}\right.$$

$$\left. + \left(\frac{\rho\xi}{2x}\right)^2 \prod_i (x-x_i^-) \prod_i (x-x_i^+)\right\} G = 0 \tag{H-11}$$

$F(x)$ and $G(x)$ have the solution of the form

$$F(x) = x^\alpha \sum_{n=0}^{\infty} A_n x^n \; , \text{ and } \; G(x) = x^\beta \sum_{n=0}^{\infty} B_n x^n \; .$$

The indicial equations are now

$$\alpha^2 - (\kappa^2 - (\xi\rho)^2) = 0 \qquad \text{for} \quad F(x) ,$$

$$\beta^2 - (\kappa^2 - (\xi\rho)^2) = 0 \qquad \text{for} \quad G(x) .$$

We thus have

$$\alpha = \pm\gamma , \quad \text{and} \qquad \beta = \pm\gamma$$

where

$$\gamma^2 = \kappa^2 - (\xi\rho)^2 = \kappa^2 - \left(\frac{\alpha Z}{2}\right)^2 . \tag{H-12}$$

In order to determine the ratio F/G, we may write F(r) and G(r) as

$$F(r) = F_0 r^\gamma \sum A_n r^n = F_0 r^\gamma \sum a_n , \tag{H-13}$$

$$G(r) = G_0 r^\gamma \sum B_n r^n = G_0 r^\gamma \sum b_n , \tag{H-14}$$

where F_0 and G_0 are constants, the ratio of which are going to be determined. Substituting eqs.(H-13) and (H-14) into (H-1) and (H-2), respectively, and comparing the coefficients of the powers of r, we obtain the following equations:

$$F_0(\kappa-\gamma)A_0 = G_0\xi\rho B_0 , \tag{H-15a}$$

$$G_0(\gamma+\kappa)B_0 = F_0\xi\rho A_0 , \tag{H-15b}$$

$$F_0(\kappa-\gamma-n)A_n = \left\{(W-1 + \frac{3\xi}{2})B_{n-1} + \xi\rho B_n - \frac{\xi}{2\rho^2} B_{n-3}\right\}G_0 , \tag{H-15c}$$

$$G_0(\kappa+\gamma+n)B_n = \left\{(W+1 + \frac{3\xi}{2})A_{n-1} + \xi\rho A_n - \frac{\xi}{2\rho^2} A_{n-3}\right\}F_0 . \tag{H-15d}$$

From eqs.(H-15a) and (H-15b), we again obtain the same expression $\gamma^2 = \kappa^2 - (\xi\rho)^2$. (See eq.(H-12)) We now choose F_0 and G_0 so as to make A_0 and B_0 be equal to 1. Thus we have

$$\frac{F_0}{G_0} = \frac{\xi\rho}{\kappa-\gamma} \quad \text{or} \quad \frac{\kappa+\gamma}{\xi\rho} \tag{H-16}$$

and
$$a_0 = b_0 = A_0 = B_0 = 1 \ . \tag{H-17}$$

What we should do next is to determine the inter-relation between the coefficents A_n and B_n. This can be facilitated by using eq.(H-15c).

$$F_0(\kappa-\gamma-n)r^nA_n = G_0\left[(W-1+\frac{3\xi}{2})r^nB_{n-1} + \xi\rho\, r^nB_n - \frac{\xi}{2\rho^2}r^nB_{n-3}\right].$$

Thus we have

$$F_0(\kappa-\gamma-n)a_n = G_0\left[(W-1+\frac{3\xi}{2})r\, b_{n-1} + \xi\rho\, b_n - \frac{\xi r^3}{2\rho^2}b_{n-3}\right]$$

which leads to the expression

$$F_0(\kappa-\gamma-n)a_n - G_0\xi\rho b_n = G_0\left[r(W-1+\frac{3\xi}{2})b_{n-1} - \frac{\xi r^3}{2\rho^2}b_{n-3}\right].$$

Similarly, we have from eq.(H-15d),

$$F_0\xi\rho a_n - G_0(\kappa+\gamma+n)b_n = F_0\left[-r(W+1+\frac{3\xi}{2})a_{n-1} + \frac{\xi r^3}{2\rho^2}a_{n-3}\right].$$

These two equations are now combined in the matrix form

$$\begin{pmatrix} (\kappa-\gamma-n)F_0 & -\xi\rho G_0 \\ \xi\rho F_0 & (\kappa+\gamma+n)G_0 \end{pmatrix} \begin{pmatrix} a_n \\ b_n \end{pmatrix} = \begin{pmatrix} r(W-1+\frac{3\xi}{2})b_{n-1} & -\frac{\xi r^3}{2\rho^2}b_{n-3} \\ -r(W+1+\frac{3\xi}{2})a_{n-1} & \frac{\xi r^3}{2\rho^2}a_{n-3} \end{pmatrix} .$$

We may thus obtain

$$a_n = -\frac{\alpha Zr}{2}\frac{1}{n(n+2\gamma)}\Big[(1+\frac{n}{\kappa+\gamma})\Big\{(W-1+\frac{3\xi}{2})b_{n-1} - \frac{\xi r^2}{2\rho^2}b_{n-3}\Big\}$$

$$+ (W+1+\frac{3\xi}{2})a_{n-1} - \frac{\xi r^2}{2\rho^2}a_{n-3}\Big] . \tag{H-18}$$

$$b_n = -\frac{\alpha Zr}{2}\frac{1}{n(n+2\gamma)}\Big[(1-\frac{n}{\kappa-\gamma})\Big\{(W+1+\frac{3\xi}{2})a_{n-1} - \frac{\xi r^2}{2\rho^2}a_{n-3}\Big\}$$

$$+ (W-1+\frac{3\xi}{2})b_{n-1} - \frac{\xi r^2}{2\rho^2}b_{n-3}\Big] . \tag{H-19}$$

Thus we have obtained the recurrence relations satisfied by a_n and b_n for the potential specified by eq.(H-3). It is apparent from eqs.(H-18) and (H-19) that if the ratio F/G is evaluated at the nuclear surface $r = \rho$, a's and b's become

$$a_n = -\frac{\alpha Z\rho}{2}\frac{1}{n(n+2\gamma)}\Big[(1+\frac{n}{\kappa+\gamma})\Big\{(W-1+\frac{3\xi}{2})b_{n-1} - \frac{\xi}{2}b_{n-3}\Big\}$$

$$+ \quad (W+1+\frac{3\xi}{2})a_{n-1} - \frac{\xi}{2}a_{n-3}\Big],$$

(H-20)

and

$$b_n = -\frac{\alpha Z \rho}{2}\frac{1}{n(n+2\gamma)}\Big[(1-\frac{n}{\kappa-\gamma})\{(W+1+\frac{3\xi}{2})a_{n-1} - \frac{\xi}{2}a_{n-3}\}$$

$$+ \quad (W-1+\frac{3\xi}{2})b_{n-1} - \frac{\xi}{2}b_{n-3}\Big].$$

(H-21)

APPENDIX I

ROOT MEAN SQUARE RADIUS FOR DIFFUSED CHARGE DISTRIBUTION

A root mean square radius $\sqrt{\langle r^2 \rangle}$ can be calculated from

$$\langle r^2 \rangle = \int \rho_e(r)\, r^2 d\vec{r} = 4\pi \int_0^{\infty} \rho_e(r)\, r^4 dr \qquad \text{(I-1)}$$

where the charge distribution $\rho_e(r)$ is written as

$$\rho_e(r) = \rho_e^0 \qquad (0 \leq r \leq \rho - t),$$

$$\rho_e(r) = \frac{\rho_e^0}{2}\left(1 + \sinh\frac{\rho - r}{a\rho}\right) \qquad (\rho - t \leq r \leq \rho + t),$$

$$\rho_e(r) = 0 \qquad (\rho + t < r < \infty).$$

Thus $\langle r^2 \rangle$ may be written in the following form;

$$\langle r^2 \rangle = 4\pi \frac{\rho_e^0}{2}\left[\int_0^{\rho - t} 2r^4 dr + \int_{\rho - t}^{\rho + t} r^4 dr + \int_{\rho - t}^{\rho + t} \sinh\frac{\rho - r}{a\rho} r^4 dr\right]$$

$$= 4 \frac{\rho_e^0}{2}\left[\frac{1}{5}\left\{(\rho - t)^5 + (\rho + t)^5\right\} + \int_{\rho - t}^{\rho + t} \sinh\frac{\rho - r}{a\rho} r^4 dr\right] \qquad \text{(I-2)}$$

The integration of the right hand side of eq.(I-2) can be calculated straightforwardly to yield

$$\int_{\rho-t}^{\rho+t} \sinh\frac{\rho-r}{a\rho}\, r^4 dr$$

$$= 8a^2\rho^5 \left[1+3\left(\frac{t}{\rho}\right)^2 - 6a^2\left(\sqrt{2}\frac{t}{a\rho} - 1\right) - \sqrt{2}\frac{t}{a\rho}\left\{1+\left(\frac{t}{\rho}\right)^2\right\}\right] . \tag{I-3}$$

Substituting eq.(I-3) into eq.(I-2) with the help of the normalization condition

$$4\pi \int_0^\infty \rho_e(r) r^2 dr = 1 \quad ,$$

the root mean square radius $\langle r^2 \rangle$ can be readily obtained as

$$\langle r^2 \rangle = \frac{3}{5}\rho^2 \left[\frac{1+10v^2\left\{1+\frac{1}{2}v^2 - \frac{2\sqrt{2}}{s}(1+v^2)\right\}}{1+6v^2\left(\frac{1}{s}-\frac{1}{\sqrt{2}}\right)^2} + 20\frac{v^2}{s^2} \right]$$

where $\quad v = \frac{t}{\rho} \quad$ and $\quad s = \sinh^{-1}(1)$.

Related to this r.m.s radius are the radius of the equivalent uniform distribution, which gives the same value of $\langle r^2 \rangle$ as the actual distribution does, and the radial constant r_0. The equivalent uniform radius, R, is given by

$$\langle r^2 \rangle = \frac{3}{5} R^2$$

and the radial constant r_0 is defined by

$$r_0 = \frac{R}{A^{1/3}} \quad .$$

APPENDIX J

RECURRENÇE RELATIONS

FOR THE LEGENDRE FUNCTIONS

The recurrence relations for our present purpose are the following.

$$P^{m}_{\ell+1}(x) - xP^{m}_{\ell}(x) - (\ell+m)(1-x)^{\frac{1}{2}}P^{m-1}_{\ell}(x) = 0 \qquad (J-1)$$

$$xP^{m}_{\ell}(x) - (\ell-m+1)(1-x)^{\frac{1}{2}}P^{m-1}_{\ell}(x) - P^{m}_{\ell-1}(x) = 0. \qquad (J-2)$$

Let m and ℓ be 1 and $j-\frac{1}{2}$ respectively in eq.(J-1). We then have

$$P^{1}_{j+\frac{1}{2}} - \cos\theta\, P^{1}_{j-\frac{1}{2}} - (j+\tfrac{1}{2})\sin\theta\, P_{j-\frac{1}{2}} = 0 \quad . \qquad (J-3)$$

In eq.(J-2), let m and ℓ be 1 and $j+\frac{1}{2}$ respectively and we obtain

$$\cos\theta\, P^{1}_{j+\frac{1}{2}} - (j+\tfrac{1}{2})\sin\theta\, P_{j+\frac{1}{2}} - P^{1}_{j-\frac{1}{2}} = 0 \quad ,$$

or

$$(j+\tfrac{1}{2})\sin\theta = \cos\theta\, P^{1}_{j+\frac{1}{2}} - P^{1}_{j-\frac{1}{2}} \quad . \qquad (J-4)$$

Subtracting eq.(J-4) multipled by $\cos\theta$ from eq.(J-3), we have

$$\sin\theta\, P^{1}_{j+\frac{1}{2}} = (j+\tfrac{1}{2})\, P_{j-\frac{1}{2}} - (j+\tfrac{1}{2})\cos\theta\, P_{j+\frac{1}{2}} \quad . \qquad (J-5)$$

Furthermore using eqs.(J-3) and (J-4), we have

$$P^1_{j+\frac{1}{2}} - P^1_{j-\frac{1}{2}} = (j+\tfrac{1}{2}) \frac{\sin\frac{\theta}{2}}{\cos\frac{\theta}{2}} \left(P_{j+\frac{1}{2}} + P_{j-\frac{1}{2}} \right) , \qquad \text{(J-6)}$$

$$P^1_{j+\frac{1}{2}} + P^1_{j-\frac{1}{2}} = - (j+\tfrac{1}{2}) \frac{\cos\frac{\theta}{2}}{\sin\frac{\theta}{2}} \left(P_{j+\frac{1}{2}} - P_{j-\frac{1}{2}} \right) . \qquad \text{(J-7)}$$

It is now ready to calculate $(\chi^1_j + \chi^2_j)$ and $(\chi^1_j - \chi^2_j)$, where

$$\chi^1_j + \chi^2_j = (j+\tfrac{1}{2})\left\{P_{j+\frac{1}{2}} + P_{j-\frac{1}{2}}\right\}\alpha + \left\{P^1_{j+\frac{1}{2}} - P^1_{j-\frac{1}{2}}\right\}e^{i\varphi}\beta \ .$$

Using eq.(J-6), we can write as

$$\chi^1_j + \chi^2_j = (j+\tfrac{1}{2})\left\{P_{j+\frac{1}{2}} + P_{j-\frac{1}{2}}\right\}\sec\frac{\theta}{2}\left[\cos\frac{\theta}{2}\alpha + \sin\frac{\theta}{2}e^{i\varphi}\beta\right] . \qquad \text{(J-8)}$$

Similary, we find, using eq.(J-7) that

$$\chi^1_j - \chi^2_j = (j+\tfrac{1}{2})\left\{P_{j+\frac{1}{2}} - P_{j-\frac{1}{2}}\right\}\operatorname{cosec}\frac{\theta}{2}e^{i\varphi}\left[-\sin\frac{\theta}{2}e^{-i\varphi}\alpha + \cos\frac{\theta}{2}\beta\right] . \qquad \text{(J-9)}$$

We now prove one of the relations written in the text

$$\vec{\sigma}\cdot\vec{r}\,\chi^2_j = r\,\chi^1_j \ .$$

To this end, we write

$$\vec{\sigma}\cdot\vec{r}\,\chi^2_j = r\vec{\sigma}\cdot\hat{r}\,\chi^2_j$$

and

$$\vec{\sigma}\cdot\hat{r} = \begin{pmatrix} \cos\theta & \sin\theta\, e^{-i\varphi} \\ \sin\theta\, e^{i\varphi} & -\cos\theta \end{pmatrix} .$$

Thus

$$\vec{\sigma}\cdot\vec{r}\,\chi_j^2 = r\begin{pmatrix} \cos\theta & \sin\theta\, e^{-i\varphi} \\ \sin\theta\, e^{i} & -\cos\theta \end{pmatrix}\begin{pmatrix} (j+\frac{1}{2})P_{j+\frac{1}{2}} \\ e^{i\varphi}P_{j+\frac{1}{2},1} \end{pmatrix}$$

$$= r\begin{pmatrix} (j+\frac{1}{2})\cos\theta P_{j+\frac{1}{2}} + \sin\theta\, P_{j+\frac{1}{2},1} \\ (j+\frac{1}{2})e^{i\varphi}\sin\theta\, P_{j+\frac{1}{2}} - e^{i\varphi}\cos\theta\, P_{j+\frac{1}{2},1} \end{pmatrix} ,$$

and using eqs.(J-4) and (J-5), this yields that

$$\vec{\sigma}\cdot\vec{r}\,\chi_j^2 = r\begin{pmatrix} (j+\frac{1}{2})\, P_{j-\frac{1}{2}} \\ -\, e^{i\varphi}P_{j-\frac{1}{2},1} \end{pmatrix} = r\,\chi_j^1 .$$

Similarly, we may find that

$$\vec{\sigma}\cdot\vec{r}\,\chi_j^1 = r\,\chi_j^2 .$$

Further it follows from equation $2\vec{L}\cdot\vec{S}\,\chi_j = \lambda\hbar^2\chi_j$ that

$$\vec{L}\cdot\vec{\sigma}\chi_j = \lambda\,\hbar\chi_j$$

where

$$\lambda = \frac{\vec{J}^2}{\hbar^2} - \ell(\ell+1) - \frac{3}{4} .$$

If χ_j^1 is considered, this means $\lambda = \ell$ and $\vec{J}^2 = \hbar^2(\ell+\frac{1}{2})(\ell+3/2)$, as we have seen, hence we have

$$\vec{\sigma}\cdot\vec{L}\,\chi_j^1 = \hbar\,\ell\,\chi_j^1 = \hbar(j-\tfrac{1}{2})\,\chi_j^1 \quad .$$

For χ_j^2, since $\lambda = -\ell - 1$ and $\vec{J}^2 = \hbar^2(\ell-\frac{1}{2})(\ell+\frac{1}{2})$, we have

$$\vec{\sigma}\cdot\vec{L}\,\chi_j^2 = -\hbar(\ell+1)\,\chi_j^2 = -\,\hbar(j+3/2)\,\chi_j^2 \quad .$$

APPENDIX K

ALTERNATIVE METHOD TO DETERMINE η IN THE HIGH ENERGY LIMIT

We, as usual, start with the basic equation

$$e^{2i\eta} = - \frac{\kappa - iy/W}{\gamma + iy} \tag{K-1}$$

and

$$\tan 2\eta = - i \frac{e^{i2\eta} - e^{-i2\eta}}{e^{i2\eta} + e^{-i2\eta}} \tag{K-2}$$

where $\quad \gamma^2 = \kappa^2 - (\alpha Z)^2 \quad$ and $\quad y = \alpha ZW/p$.

From eq.(K-1), we observe

$$\left|\kappa - iy/W\right| = \left|\gamma + iy\right|$$

since the magnitute of the $e^{2i\eta}$ is equal to 1. Thus we obtain the relation

$$\kappa^2 + y^2/W^2 = \gamma^2 + y^2 . \tag{K-3}$$

Substituting eq.(K-1) into eq.(K-2) with eq.(K-3), we have

$$\tan 2\eta = i \frac{(\gamma+iy)^2 - (\kappa-iy/W)^2}{(\gamma+iy)^2 + (\kappa-iy/W)^2} = - \frac{y(\kappa+\gamma/W)}{\kappa\gamma - y^2/W} .$$

Thus we obtain

$$\eta = - \tfrac{1}{2}\tan^{-1} \frac{y(\kappa+\gamma/W)}{\kappa\gamma - y^2/W} .$$

In the high energy limit, where $y \longrightarrow \alpha Z$,

$$\eta = -\tfrac{1}{2}\tan^{-1}\frac{\alpha Z}{\gamma} \tag{K-4}$$

whereas, in the BR expression, in the high energy limit

$$\tan\eta_{BR} = \frac{\alpha Z}{p}\,\frac{W-1}{\kappa-\gamma} \longrightarrow \frac{\alpha Z}{\kappa-\gamma} \tag{K-5}$$

where we put a subscript BR for specifying η of BR. Using the relation,

$$\tan 2\eta = \frac{2\tan\eta}{1-\tan^2\eta} \tag{K-6}$$

and substituting eq.(K-5) into the right hand side of eq.(K-6), we have as an approximate form

$$\frac{2\tan\eta_{BR}}{1-\tan^2\eta_{BR}} = \frac{2y(\kappa-\gamma)}{(\kappa-\gamma)^2-y^2} = -\frac{\alpha Z}{\gamma} \quad . \tag{K-7}$$

We, therefore, obtain from eqs.(K-6) and (K-7)

$$\eta_{BR} = -\tfrac{1}{2}\tan^{-1}\frac{\alpha Z}{\gamma} \quad . \tag{K-8}$$

Comparing eq.(K-8) with (K-4), we know that both η approach the same value.

APPENDIX L

COULOMB PHASE SHIFT

FOR THE IRREGULAR SOLUTION

The difference of the Coulomb phase shifts between the regular and irregular solutions is given by YRW in the following form

$$e^{i(\bar{\delta}^c_j - \delta^c_j)} = \frac{1-i\tan\pi(k-\gamma)\coth\pi y}{|1-i\tan\pi(k-\gamma)\coth\pi y|}\, e^{-\pi i(k-\gamma)} \qquad \text{(L-1)}$$

which can be written as

$$e^{i(\bar{\delta}^c_j - \delta^c_j)} = e^{i\arg(1-i\tan\pi(k-\gamma)\coth\pi y)}\, e^{-\pi i(k-\gamma)} . \qquad \text{(L-2)}$$

If we define a complex number z and x as

$$z = x + iy , \qquad x = k - \gamma$$

then we have

$$\sin\pi z = \sin\pi x\cdot\cos(i\pi y) + \cos\pi x\cdot\sin(i\pi y)$$

$$= \sin\pi x\cdot\cosh\pi y + i\cos\pi x\cdot\sinh\pi y .$$

Therefore we obtain

$$\sin\pi(k-\gamma+iy) = i\cos\pi(k-\gamma)\sin\pi y\left\{1-i\tan\pi(k-\gamma)\coth\pi y\right\} \qquad \text{(L-3a)}$$

or $$\sin\pi(k-\gamma+iy) = \cos\pi(k-\gamma)\sin\pi y\left\{i+\tan\pi(k-\gamma)\coth\pi y\right\} . \qquad \text{(L-3b)}$$

From eq.(L-3a), it is found that

$$\arg \sin\pi(k-\gamma+iy) = \arg i + \arg\{1-i\tan\pi(k-\gamma)\coth\pi y\}$$

$$= \arg\{1-i\tan\pi(k-\gamma)\coth\pi y\} + \frac{\pi}{2} \quad .(\text{L-4})$$

Substituting eq.(L-4) into eq.(L-2), we find that

$$e^{i(\bar{\delta}_j^c - \delta_j^c)} = e^{i \arg \sin\pi(k-\gamma+iy)} \, e^{-\frac{\pi}{2}i} \, e^{-\pi i(k-\gamma)} \quad . \qquad (\text{L-5})$$

It is, however, found that from eq.(L-3b)

$$\arg \sin\pi(k-\gamma+iy) = \tan^{-1}\{\tanh\pi y \cot\pi(k-\gamma)\}. \qquad (\text{L-6})$$

using the general expression for a complex number z

$$\tan^{-1} z = \frac{1}{2i} \ln \frac{1+iz}{1-iz} \; .$$

A part of the right hand side of eq.(L-5) can be rewritten, with the help of eq.(L-6), as

$$e^{i \arg \sin\pi(k-\gamma+iy)}$$

$$= e^{\ln\{(1+i\cot\pi(k-\gamma)\tanh\pi y)/(1-i\cot\pi(k-\gamma)\tanh\pi y)\}^{\frac{1}{2}}}$$

$$= \left\{\frac{1+i\cot\pi(k-\gamma)\tanh\pi y}{1-i\cot\pi(k-\gamma)\tanh\pi y}\right\}^{\frac{1}{2}}$$

$$= \left\{\frac{\sin\pi(k-\gamma)\cosh\pi y + i\cos\pi(k-\gamma)\sinh\pi y}{\sin\pi(k-\gamma)\cosh\pi y - i\cos\pi(k-\gamma)\sinh\pi y}\right\}^{\frac{1}{2}}$$

$$= \left\{\frac{\sin\pi(k-\gamma+iy)}{\sin\pi(k-\gamma-iy)}\right\}^{\frac{1}{2}} = \left[\frac{\dfrac{\pi}{\sin\pi(k-\gamma-iy)}}{\dfrac{\pi}{\sin\pi(k-\gamma+iy)}}\right]^{\frac{1}{2}} \qquad (\text{L-7})$$

Now, using the properties of Gamma function

$$\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin\pi z} \quad \text{and} \quad \Gamma(z) = (z-1)\Gamma(z-1),$$

eq.(L-7) can be written

$$\left[\frac{\dfrac{\pi}{\sin\pi(k-\gamma-iy)}}{\dfrac{\pi}{\sin\pi(k-\gamma+iy)}}\right]^{\frac{1}{2}} = \left\{\frac{\Gamma(k-\gamma-iy)\Gamma(1-k+\gamma+iy)}{\Gamma(k-\gamma+iy)\Gamma(1-k+\gamma-iy)}\right\}^{\frac{1}{2}}$$

$$= \left\{\frac{\Gamma(k-1-\gamma-iy)}{\Gamma(k-1-\gamma+iy)}\,\frac{(1-k+\gamma+iy)}{(1-k+\gamma-iy)}\,\frac{\Gamma(1-k+\gamma+iy)}{\Gamma(1-k+\gamma-iy)}\right\}^{\frac{1}{2}}$$

$$= \left\{\frac{\Gamma(k-1-\gamma-iy)\,\Gamma(1-(k-1)+\gamma+iy)}{\Gamma(k-1-\gamma+iy)\,\Gamma(1-(k-1)+\gamma-iy)}\right\}^{\frac{1}{2}} = \text{------}$$

$$= \left\{\frac{\Gamma(k-n-\gamma-iy)\,\Gamma(1-(k-n)+\gamma+iy)}{\Gamma(k-n-\gamma+iy)\,\Gamma(1-(k-n)+\gamma-iy)}\right\}^{\frac{1}{2}} = \text{------}$$

$$= \left\{\frac{\Gamma(1-\gamma-iy)\,\Gamma(\gamma+iy)}{\Gamma(1-\gamma+iy)\,\Gamma(\gamma-iy)}\right\}^{\frac{1}{2}} = \left\{\frac{\sin\pi(\gamma-iy)}{\sin\pi(\gamma+iy)}\right\}^{\frac{1}{2}}.$$

Therefore we find that

$$e^{i\,\arg\,\sin\pi(k-\gamma+iy)} = \left\{\frac{\sin\pi(\gamma-iy)}{\sin\pi(\gamma+iy)}\right\}^{\frac{1}{2}}. \tag{L-8}$$

Substituting eq.(L-8) into eq.(L-5), we obtain

$$e^{i(\bar{\delta}_j^{c'} - \delta_j^{c})} = e^{-\pi i(k-\gamma)}\, e^{-\frac{\pi}{2}i} \left\{\frac{\sin\pi(\gamma-iy)}{\sin\pi(\gamma+iy)}\right\}^{\frac{1}{2}}. \tag{L-9}$$

Taking the square of eq.(L-9), we find

$$e^{2i\bar{\delta}_j^c} = e^{2i\delta_j^c}\, e^{-2\pi i(k-\gamma)}\, e^{-\pi i}\, \frac{\sin\pi(\gamma-iy)}{\sin\pi(\gamma+iy)}$$

which reduces, by using eq.(7-3) in the text, to

$$e^{2i\bar{\delta}_j^c} = e^{-\pi i(k-\gamma)}\, e^{-\pi i}\, \frac{\gamma-iy}{k}\, \frac{\sin\pi(\gamma-iy)}{\sin\pi(\gamma+iy)}\, \frac{\Gamma(\gamma-iy)}{\Gamma(\gamma+iy)}$$

$$= e^{-\pi i(k-\gamma)}\, e^{-\pi i}\, \frac{\gamma-iy}{k}\, \frac{\Gamma(1-(\gamma+iy))}{\Gamma(1-(\gamma-iy))}$$

$$= e^{-\pi i(k-\gamma)}\, e^{-\pi i}\, \frac{\gamma+iy}{k}\, \frac{\Gamma(-\gamma-iy)}{\Gamma(-\gamma+iy)}$$

$$= e^{2i\{-\arg\Gamma(-\gamma+iy) + \frac{\pi}{2}\gamma - \frac{\pi k}{2}\}}\, e^{-\pi i}\, \frac{\gamma+iy}{k}\ .$$

If $\kappa<0$, using eq.(7-6c), we find

$$e^{2i\bar{\delta}_j^c} = e^{2i\{-\arg\Gamma(-\gamma+iy) + \frac{\pi\gamma}{2} - \frac{\pi k}{2} + \bar{\eta}_\kappa\}}\ .$$

We thus have

$$\bar{\delta}_j^c = -\arg\Gamma(-\gamma+iy) + \frac{\pi\gamma}{2} + \bar{\eta}_\kappa - \frac{\pi}{2}(j+\tfrac{1}{2})$$

$$= -\arg\Gamma(-\gamma+iy) + \frac{\pi\gamma}{2} + \bar{\eta}_\kappa - \frac{\pi}{2}(\ell+1) \quad \text{for } \kappa<0\ . \tag{L-10}$$

Similarly, if $\kappa>0$, using eq.(7-6d), we find

$$e^{2i\bar{\delta}_j^c} = e^{2i\{-\arg\Gamma(-\gamma+iy) + \frac{\pi\gamma}{2} - \frac{\pi}{2}(k+1) + \bar{\eta}_\kappa\}}\ .$$

We thus have for $\kappa > 0$

$$\bar{\delta}_j^c = -\arg\Gamma(-\gamma+iy) + \frac{\pi\gamma}{2} - \frac{\pi}{2}(k+1) + \bar{\eta}_\kappa \quad . \qquad \text{(L-11)}$$

BIBLIOGRAPHY

1) M. E. Rose and D. K. Holmes, Phys. Rev. <u>83</u>, 190 (1951).

2) M. E. Rose and C. L. Perry, Phys. Rev. <u>90</u>, 479 (1953).

3) This effect is not considered in this work. See, for example W. Bühring, Nucl. Phys. <u>40</u>, 472 (1963); <u>65</u>, 369 (1965).

4) C. P. Bhalla and M. E. Rose, Oak Ridge National Laboratory Report ORNL-2954 (1960) and ORNL-3207 (1961).

5) C. P. Bhalla and M. E. Rose, Phys. Rev. <u>128</u>, 774 (1962).

6) J. N. Huffaker and C. E. Laird, Nucl. Phys. <u>A92</u>, 584 (1967).

7) W. Bühring, Report, Kernforschungszentrum Karlsruhe, KFK559 (April 1967).

8) M. E. Rose, Elementary Theory of Angular Momentum (John Wiley & Sons, Inc., New York, 1961).

9) M. A. Preston , Physics of the Nucleus (Addison-Wesley, Reading, Mass., 1962).

10) B. N. Taylor, W. H. Parker and D. N. Langenberg, Rev. of Mod. Phys. <u>41</u>, 375 (1969).

11) M. E. Rose, Relativistic Electron Theory (John Wiley & Sons, Inc., New York, 1961).

12) E. T. Whittaker and G. N. Watson, A Course of Modern Analysis (Cambridge University Press, 1927).

13) I. N. Sneddon, Special Functions of Mathematical Physics and Chemistry (Oliver and Boyd, London, 1956).

14) J. Mathews and R. L. Walker, Mathematical Methods of Physics (W. A. Benjamin, New York, second edition 1970).

15) See, for example, D. F. Jackson, Nuclear Reactions (Methuen, London, 1970).

16) S. S. Schweber, An Introduction to Relativistic Quantum Field Theory (Harper and Row, Inc., New York, 1962).

17) F. Mandl, Introduction to Quantum Field Theory (Interscience Publishers Inc., New York, 1966).

18) C. S. Wu and S. A. Moszkowski, Beta Decay (Interscience Publishers Inc., New York, 1966).

19) E. J. Konopinski and G. E. Uhlenbeck, Phys. Rev. 60, 308 (1941).

20) M. Morita, Prog. Theoret. Phys. (Kyoto), Suppl. 26 (1963).

21) D. H. Wilkinson, Nucl. Phys. A 158, 476 (1970).

22) H. Ogata and J. Asai, Phys. Lett. 48B, 91 (1974).

23) H. Behrens and W. Bühring, Nucl. Phys. A150, 481 (1970).

24) H. Behrens and W. Bühring, Nucl. Phys. A162, 111 (1971).

25) R. Hofstadter ed., Nuclear and Nucleon Structure (Benjamin, New York, 1963), which contains most of the important papers concerning electron scattering up to 1962.

26) L. K. Acheson, Jr., Phys. Rev. 82, 488 (1951).

27) H. A. Bethe and F. deHoffmann, Mesons and Fields, Vol 2 (Row, Peterson, New York, 1955).

28) R. E. Marshak, Meson Physics (McGraw-Hill, New York, 1952).

29) D. R. Yennie, D. G. Ravenhall and R. N. Wilson, Phys. Rev. 95, 500 (1954).

30) A. R. Edmonds, Angular Momentum in Quantum Mechanics (Princeton University Press, Princeton, New Jersey, 1960).

31) E. Merzbacher, Quantum Mechanics (John Wiley & Sons, New York, 1961).

32) H. Überall, Electron Scattering from Complex Nuclei (Academic Press, New York, 1971).

33) M. Croissiaux, R. Hofstadter, A. E. Walker, M. R. Yearian, D. G. Ravenhall, B. C. Clark and R. Herman, Phys. Rev. 137, B865 (1965).

34) J. Alster, B. F. Gibson, J. S. McCarthy, M. S. Weiss and R. M. Wright, Phys. Rev. C7, 1089 (1973).

VITA AUCTORIS

The author was born on October 19, 1943 in Tokyo, Japan. In 1963 he enrolled in the physics programme at Sophia University and received his B.Sc. in 1967 and subsequently entered the Graduate School of Sophia University. In 1969 he received the degree of M.Sc. on the thesis titled "Panofsky Ratio in Vector Meson Dominance Model" and "Chiral Dynamics, A Review" under the supervision of Dr. A. Fujii.

In 1969 he came to Canada and entered the Graduate Studies of University of Windsor for his Ph.D. under the supervision of Dr. H. Ogata. His research project consists of two parts, one the pairing model calculation, and the other beta decay and high energy electron scattering.